A constrained robust model predictive control using modified fuzzy disturbance observer for continuous-time systems with polytopic uncertainty

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Abstract: In this paper, a constrained robust model predictive control (MPC) is addressed for continuous-time systems with polytopic uncertainty. A modified fuzzy disturbance observer (MFDO) is applied to estimate the uncertainty at each vertex given by a linear model. The quantity of uncertainty estimated by the MFDO is utilized as a pseudo membership grade for each vertex. The state feedback MPC control law is constructed by the weighted sum of each control gain according to the pseudo membership grade. To illustrate the efficiency of the proposed method, numerical simulation is represented.

Keywords: Robust model predictive control, Polytopic uncertainty, Fuzzy disturbance observer.

1. INTRODUCTION

Model predictive control (MPC) has attracted lots of attention to study in the last decade. Especially due to the ability to handle inequality constraints on the manipulated and controlled variables, MPC techniques are widely used in industrial process control practice. In standard MPC algorithm, the optimal control input is computed at each time instance by solving optimization problem over a fixed time horizon. One of the main drawbacks of MPC is the difficulty to incorporate plant model uncertainties explicitly. MPC uses estimated state based on approximation of model, which is not a real model, to calculate the optimal cost. Thus it is important for MPC to be robust to model uncertainty. To solve this problem, robust MPC techniques have been investigated [1]–[3]. Polytopic system description is a representative tool to design controllers for uncertain systems and nonlinear systems as well. In this description the plant is considered as a linear time varying (LTV) system which is represented by convex combination of each vertex of the polytope. For polytopic systems, Kothare et al. used a fixed Lyapunov function [1], [2] and Cuzzola et al. proposed polytope-dependent Lyapunov function [3]. Since the uncertainties are not measurable, in these methods, the feedback gain is fixed for each sampling time instance in order to satisfy the stability condition within the polytope. If the uncertainties can be measured, one can obtain a less conservative condition for uncertain or nonlinear systems.

In order to estimate uncertainties, the fuzzy disturbance observer (FDO) was proposed by Kim [4]. Based on the adaptive fuzzy control scheme, the FDO is used to compensate for the external disturbance and internal parameter perturbation. In the conventional FDOB, it is assumed that the disturbance is constant or varies slowly. If the external disturbance varies fast, the conventional FDOB may fail to recover the closed loop performance of the nominal system. Therefore, it is desired to modify the conventional FDO to achieve high precision even in the presence of fast disturbance.

In this paper, we propose a robust constrained MPC for continuous-time systems with polytopic uncertainty. The uncertainty is estimated by the FDO and the FDO is modified to achieve high precision and reliability. The quantity of the estimated uncertainty is utilized as a pseudo membership grade for each vertex of the polytope. Then, the robust MPC control law is obtained in the form of parallel distributed compensation (PDC) which gives less conservative result than using fixed gain. The closed-loop stability is analyzed based on Lyapunov method using linear matrix inequality (LMI). Also, numerical examples are represented to demonstrate the effectiveness of the proposed method.

2. MODIFIED FUZZY DISTURBANCE OBSERVER

In order to overcome the drawback of the conventional FDO, we modify it to have more robust performance. By dealing with the disturbance reconstruction error in the adaptation law, the fuzzy parameter vector of the modified FDO (MFDO) converges to its optimal value faster than the conventional FDO.

Let us consider the following system:

\[ \dot{x} = (f(x) + \Delta f(x)) + (g(x) + \Delta g(x))u + d \]

\[ = f(x) + g(x)u + \Omega \]  

(1)

where \( x \in \mathbb{R} \) is the state of the system, \( f(x) \) and \( g(x) \) are the known nominal functions, \( \Delta f(x) \) and \( \Delta g(x) \) are their respective uncertainty, \( d \) is unknown external disturbance, and

\[ \Omega = \Delta f(x) + \Delta g(x) + d \]  

(2)

is the total disturbance composed of all the disturbances and uncertainties. To construct the FDO \( \hat{\Omega}(x, u[\theta]) \) which is the approximation of \( \Omega(x, u) \), the following dynamic system is considered:

\[ \dot{\eta} = f(x) + g(x)u + \hat{\Omega}(x, u[\theta]) + \sigma(x - \eta) \]  

(3)
where $\sigma > 0$ and the disturbance observation error is defined as
$$\varphi = x - \eta.$$ (4)

From (1) and (3), the dynamics of disturbance observation error is obtained as
$$\dot{\varphi} = -\sigma \varphi + \Omega(x, u) - \hat{\Omega}(x, u|\theta).$$ (5)

In order to deal with the disturbance reconstruction error, let us define an identification model as follow:
$$\dot{x}_n = f(x) + g(x)u + \hat{\Omega}(x, u|\theta).$$ (6)

The disturbance reconstruction error can be defined:
$$\varepsilon = \dot{x} - \dot{x}_n = \Omega(x, u) - \hat{\Omega}(x, u|\theta) = \Omega(x, u) - \hat{\Omega}(x, u|\theta^*) + \hat{\Omega}(x, u|\theta^*) - \hat{\Omega}(x, u|\theta) = l + m$$ (7)

where $\hat{\Omega}(x, u|\theta^*) = \theta^T \xi(x)$, $\theta^*$ is the fuzzy parameter vector, $\xi(*)$ is the fuzzy basis function vector, and
$$l = \Omega(x, u) - \hat{\Omega}(x, u|\theta^*),$$ (8)
$$m = \hat{\Omega}(x, u|\theta^*) - \hat{\Omega}(x, u|\theta) = \bar{\theta}^T \xi(x),$$ (9)
$$\bar{\theta} = \theta^* - \theta,$$ (10)
$$\Theta^* = \arg\min_{\theta} \sup_{x, u} |\hat{\Omega}(x, u|\theta) - \Omega(x)|.$$ (11)

Then, we have the following theorem.

**Theorem 1.** Consider the uncertain nonlinear chaotic system (1). If the fuzzy parameter vector $\theta$ of the FDO $\dot{\theta} = \gamma_1 \xi(\varphi + \gamma_0 \varepsilon)$, $\gamma_0 > 0, \gamma_1 > 0,$ then the following robust performance is obtained:
$$\dot{\varphi}^2 \geq \sigma \dot{\theta} + \gamma_0 \varepsilon$$ (12)

Differentiating $V_d$ along the disturbance observation error dynamics (5) and using (7)–(10) leads to
$$\dot{V}_d = \varphi \dot{\varphi} + \frac{1}{\gamma_1} \bar{\theta}^T \bar{\theta}$$ (14)

Using the adaptation law (12), we have
$$\dot{V}_d = -\sigma \varphi^2 + \varphi l - \gamma_0 \bar{\theta}^T \xi(x)$$
$$= -\sigma \varphi^2 + \varphi l - \gamma_0 m^2 - \gamma_0 ml.$$ (16)

By the following inequalities:
$$\varphi l \leq \frac{1}{2} \sigma \varphi^2 + \frac{1}{2} \gamma_0 l^2,$$ (17)
$$ml \leq \frac{1}{2} m^2 + \frac{1}{2} l^2,$$ (18)

we can obtain
$$\dot{V}_d \leq -\frac{1}{2} \sigma \varphi^2 - \gamma_0 m^2 + \frac{1}{2} \gamma_0 l^2 + \frac{1}{2} \gamma_0 ml + \frac{1}{2} \gamma_0 l^2.$$ (19)

Integrating both side of (19) from 0 to $T$ yields
$$\frac{\sigma}{2} \int_0^T \varphi^2 dt + \frac{\gamma_0}{2} \int_0^T m^2 dt$$
$$\leq [V_d(0) - V_d(T)] + \frac{1}{2} \gamma_0 l^2 \int_0^T l^2 dt.$$ (20)

Since $V_d(T) > 0$, the above inequality is equivalent to (13). This completes the proof.

**Proof.** Let us consider the following Lyapunov function candidate:
$$V_d = \frac{1}{2} \varphi^2 + \frac{1}{2 \gamma_1} \bar{\theta}^T \bar{\theta}$$ (14)

Since $V_d(T) > 0$, the above inequality is equivalent to (13). This completes the proof.

**3. PROBLEM STATEMENT**

**3.1 System description**

Let us consider the following uncertain system:
$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \ [A(t) \ B(t)] \in \Phi$$ (21)

with the input constraint
$$\|u(t)\| \leq U_{max}$$ (22)

where $u(t) \in \mathbb{R}^n$, is the control input, $x(t) \in \mathbb{R}^n$ is the state of the plant, and $\Phi$ is the polytope
$$\Phi = C\alpha\{[A_1 B_1], [A_2 B_2], ..., [A_p B_p]\}$$ (23)

Fig. 1. The comparison of the conventional FDO and the MFDO.
3.2 Model predictive controller design

We consider the following quadratic objective function

\[ J_k = \int_0^\infty \{ x^T(kT + \tau, kT)Wx(kT + \tau, kT) + u^T(kT + \tau, kT)Ru(kT + \tau, kT) \} \, d\tau. \] (25)

In order to construct the model predictive controller, we need the following corollary.

**Corollary 1.** \( F_i^TRF_i + F_j^TRF_j \geq F_i^TRF_j + F_j^TRF_i \), where \( R > 0 \).

**Proof.** Since \( R > 0 \),

\[
(F_i - F_j)^T R (F_i - F_j) = F_i^T R F_i + F_j^T R F_j - F_i^T R F_j - F_j^T R F_i \geq 0
\]

that minimizes the upper bound \( V(x(kT)) \) on the robust performance objective function (25) at time \( kT \) is given by

\[
F(kT + \tau) = \sum_i h_i(kT + \tau) F_i
\]

where \( h_i(kT + \tau) \) is the normalized pseudo-membership grade defined by

\[
h_i(kT + \tau) = \frac{\exp(-\xi \hat{\Omega}_i)}{\sum_j \exp(-\xi \hat{\Omega}_j)},
\]

\( \hat{\Omega}_i \) is the output of the MFDO for \( i \)th node, \( \xi \) is a positive constant and \( F_i = M_i Q^{-1} \), if there exist positive definite matrices \( Q \) and \( M_i \) satisfying the following linear objective minimization problem:

\[
\min_{\gamma, Q, M_i} \gamma,
\]

\[
\begin{bmatrix}
1 \\
\gamma R^2 x^T(kT) \\
\end{bmatrix} \geq 0,
\]

\[
\begin{bmatrix}
A_i Q + B_i M_i \\
R \hat{\Omega}_i \\
\end{bmatrix} \leq 0
\]

where \( \gamma R^2 x^T(kT) \) is the upper bound on the performance objective function (25).

**Theorem 1.** Consider the uncertain system (21) at time \( kT \) and let \( x(kT) \) be the measured state. Then the state feedback matrix \( F \) in the control law

\[
u(kT + \tau) = F(kT + \tau)x(kT + \tau), \tau \geq 0
\]

(27)

that minimizes the upper bound \( V(x(kT)) \) on the robust performance objective function (25) at time \( kT \) is given by

\[
F(kT + \tau) = \sum_i h_i(kT + \tau) F_i
\]

where \( h_i(kT + \tau) \) is the normalized pseudo-membership grade defined by

\[
h_i(kT + \tau) = \frac{\exp(-\xi \hat{\Omega}_i)}{\sum_j \exp(-\xi \hat{\Omega}_j)},
\]

\( \hat{\Omega}_i \) is the output of the MFDO for \( i \)th node, \( \xi \) is a positive constant and \( F_i = M_i Q^{-1} \), if there exist positive definite matrices \( Q \) and \( M_i \) satisfying the following linear objective minimization problem:

\[
\min_{\gamma, Q, M_i} \gamma,
\]

\[
\begin{bmatrix}
1 \\
\gamma R^2 x^T(kT) \\
\end{bmatrix} \geq 0,
\]

\[
\begin{bmatrix}
A_i Q + B_i M_i \\
R \hat{\Omega}_i \\
\end{bmatrix} \leq 0
\]

\[
\begin{bmatrix}
A_i Q + B_i M_i + Q A_i^T + M_i^T B_i \\
A_j Q + B_j M_j + Q A_j^T + M_j^T B_j \\
\end{bmatrix}
\]

(33)

\[
\begin{bmatrix}
QW \hat{\Omega}_i \\
M_i^T R \hat{\Omega}_i \\
\end{bmatrix} \leq 0
\]

(34)

\[
\begin{bmatrix}
U_{i}^2 M_i^T Q \\
M_i^T Q + A_i Q + B_i M_i + Q A_i^T + M_i^T B_i \\
\end{bmatrix} \geq 0, \quad i = 1, ..., p, \quad i < j.
\]

(35)

**Proof.** For the simplicity of the notation, let \( x \) indicate \( x(kT + \tau, kT) \).

(a) **Existence:** Let us consider a quadratic function

\[
V(x(t)) = x(t)^T P x(t)
\]

(36)

with positive definite matrix \( P > 0 \). At sampling time \( kT \), suppose that \( V \) satisfies

\[
\frac{d}{dt} V(x) \leq -[x^T W x + u^T Ru].
\]

(37)

Using (21) and (24)--(28), \( \frac{d}{dt} V \) is given by

\[
\frac{d}{dt} V(x) = x^T P (A + B \sum_{j=1}^p h_j F_j)
\]

\[
+ (A + B \sum_{j=1}^p h_j F_j)^T P x
\]

\[
= \sum_{i=1}^p \sum_{j=1}^p h_i h_j x^T [P(A_i + B_i F_j)] + (A_i + B_i F_j)^T P x
\]

\[
= \sum_{i=1}^p h_i^2 x^T [G_{ii} + G_{ij}^T] x + \sum_{i=1}^p \sum_{j \neq i} h_i h_j x^T [G_{ij} + G_{ji}^T] x
\]

\[
= \sum_{i=1}^p h_i^2 x^T [G_{ii} + G_{ij}^T] x + \sum_{i=1}^p \sum_{j > i} h_i h_j x^T [G_{ij} + G_{ji}^T] x
\]

\[
+ G_{ji} + G_{ij}^T x
\]

(38)

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where $G_{ij} = P(A_i + B_i F_j)$. Using the fact $\sum_{i=1}^{p} h_i = 1$ and Corollary 1, (37) can be rewritten as

\[
\sum_{i=1}^{p} h_i^2 x^T [G_{ii} + G_{ii}^T + W + F_i^T RF_i] x
\]
\[
+ \sum_{i=1}^{p} \sum_{j=1}^{p} h_i h_j x^T [G_{ij} + G_{ij}^T + G_{ji} + G_{ji}^T]
\]
\[
+ 2W + F_i^T RF_i + F_j^T RF_j] x
\]
\[
\leq \sum_{i=1}^{p} h_i^2 x^T [G_{ii} + G_{ii}^T + W + F_i^T RF_i] x
\]
\[
+ \sum_{i=1}^{p} \sum_{j=1}^{p} h_i h_j x^T [G_{ij} + G_{ij}^T + G_{ji} + G_{ji}^T]
\]
\[
+ 2W + F_i^T RF_i + F_j^T RF_j] x
\]
\[
\leq 0
\]
(39) is satisfied if
\[
G_{ii} + G_{ii}^T + W + F_i^T RF_i \leq 0,
\]
(40)
\[
G_{ij} + G_{ij}^T + G_{ji} + G_{ji}^T + 2W + F_i^T RF_i + F_j^T RF_j.
\]
(41)

Let us define $Q = \gamma^{-1} P$ and $M_i = F_i Q$. Then pre-multiplying by $Q$ and using Schur complement, we see that (40) and (41) are equivalent to (32) and (33), respectively.

(b) Optimization: For finite $J_k$, we must have $x(\infty) = 0$ which yields $V(x(\infty)) = 0$. Therefore, integrating both side of (37) from $\tau = 0$ to $\infty$ leads to
\[
J_k \leq V(x(kT)),
\]
(42)
which gives the upper bound of $J_k$. Minimization of $V(x(kT))$ is equivalent to (30) subject to
\[
x^T(kT) P x(kT) \leq \gamma
\]
(43)
which is equivalent to (31).

(c) Input constraint: Since $u(kT + \tau) = \sum_{i=1}^{p} h_i u_i$, where $u_i = F_i x(kT + \tau)$, and $\sum_{i=1}^{p} h_i = 1$, the control input satisfies the input constraint (22) if each $u_i$ satisfies (22). Suppose that (37) and (43) hold at time $kT$. Since the right side of (37) is negative,
\[
x^T(kT + \tau) P x(kT + \tau) \leq x^T(kT) P x(kT), \quad \tau \geq 0
\]
(44)
Therefore, $x^T(kT + \tau) P x(kT + \tau) \leq \gamma, \tau \geq 0$ or equivalently,
\[
x^T(kT + \tau) Q x(kT + \tau) \leq 1, \quad \tau \geq 0.
\]
(45)
Thus, $E = \{ z^T Q^{-1} z \leq 1 \}$ is an invariant ellipsoid for the predicted states of the uncertain system. At time $kT$, consider the Euclidean norm constraint (22). All the future states of the entire horizon belong to the invariant ellipsoid, we have
\[
\max \| u_i(kT + \tau) \|^2 = \max \| M_i Q^{-1} x(kT + \tau) \|^2
\]
\[
\leq \max \| M_i Q^{-1} z \|^2
\]
\[
= \lambda_{max}(Q^{-1/2} M_i^T M_i Q^{-1/2}).
\]
(46)
By Schur complement, this is equivalent to (35).

Remark 1. In the proposed method, the precision of the pseudo membership function is important. Empirically, we adopt the exponential function such as (29) to obtain admissible precision. Furthermore, the inaccuracy of the pseudo membership function can be overcome by some technique such as [5] dealing with the uncertain premise variables.

4. NUMERICAL SIMULATION

In this section, the proposed model predictive control law is applied to the angular positioning system [1]. The system is described by
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha(t) \end{bmatrix} x + \begin{bmatrix} 0 \\ \kappa \end{bmatrix} u,
\]
\[
y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x
\]
(47)
where $x_1$ is the angle, $x_2$ is the angular velocity, $\kappa = 0.787 \pi \frac{d}{s^{-2}}, 0.1s^{-1} \leq \alpha(t) \leq 10s^{-1}$. The parameter $\alpha(t)$ is proportional to the coefficient of viscous friction in the rotating parts of the antenna and is assumed to be arbitrarily time-varying in the indicated range of variation. Since $0.1 \leq \alpha(t) \leq 10$, we let define
\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix}.
\]
(48)
For the simulation, $T = 0.1$, $U_{max} = 1, R = 0.00002$ and
\[
W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]
(49)
are used. The initial condition $x(0) = [0.05, 0]^T$ is employed. Fig. 2 shows the simulation result. (a) is the state of angular positioning system controlled by the proposed method while (b) is the result obtained by the robust MPC (RMPC) proposed in [1]. Compared with the RMPC, the proposed method shows better performance. (c) represents the control effort and it is shown that the input constraint $\|u\| < 1$ is satisfied.

Fig. 3 represents the reconstruction result by the pseudo membership function $h_i$ for the system state. In the proposed method, it is important how precise the membership function grade is. As shown in (b), the reconstruction error is within the scale of $10^{-3}$. After the state converges to the origin, the reconstruction error may have small value around zero. However, since the aim of the control is not to get the precise modeling result but to make the state converge to the origin, we can ignore the reconstruction error after the state converges to zero.

5. CONCLUSION

In this paper, a constrained robust model predictive control using the FDO is proposed for continuous-time systems with polytopic uncertainty. The FDO is modified to achieve the high precision. In order to design the PDC-type model predictive controller, the output of the
modified FDO is used as the pseudo membership function. Through the numerical simulation, the precision of the reconstruction by the pseudo membership function and the effectiveness of the proposed control method are verified.

REFERENCES


![Fig. 2 The simulation result.](image)

![Fig. 3 The reconstruction result by the pseudo membership function.](image)