Admissibility and dissipativity analysis for discrete-time singular systems with mixed time-varying delays

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Abstract
The problems of admissibility and dissipativity are studied for discrete-time singular systems with mixed time-varying delays. The mixed time delays consist of both discrete and distributed delays. By taking advantage of the delay partitioning technique, a criterion that warrants the admissibility of the considered systems is established. Based on the criterion, a dissipativity condition is also given. All the results presented depend upon not only discrete delay and distributed delay, but also depend upon the number of delay partitions. Some numerical examples are included to demonstrate the improvement and effectiveness of the proposed methods.

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1. Introduction

In the past decades, considerable attention has been paid to singular systems because of their applications in many areas such as electrical circuits, power systems, economics and other areas [1–4]. It should be pointed out that the study on singular systems is much more complicated than that for state-space systems, because it requires to consider not only stability, but also regularity and absence of impulses (for continuous singular systems) or causality (for discrete singular systems) simultaneously, while the latter two do not arise in the state-space ones [2,3]. On the other hand, time-delay is one of the main sources for causing instability and poor performances of dynamic systems [5], and thus much attention has been devoted to the study of time-delay systems. With the rapid development of the linear matrix inequality (LMI) approach, a number of important and interesting results have been developed for all kinds of time delay systems, e.g., [6–23] and the references therein.

As a special class of time delay systems, singular time delay systems have attracted much attention from the mathematics and control community. For example, the stability of continuous-time singular systems with time delay has been investigated in [24,25], where some LMI-based stability criteria have been proposed for testing whether the systems are stable, independent/dependent of the time delays. The $H_{\infty}$ control and filtering problems have been investigated in [26,27], respectively. In [28], the problem of sliding mode control (SMC) with passivity of a class of uncertain nonlinear singular time-delay systems has been studied. The dissipativity of singular systems with time-delay has been investigated in [29,30], where delay-dependent sufficient conditions have been established in terms of LMIs. The results related to discrete-time singular systems with time delay can be found in [31–34], where constant delay and time-varying delay have been studied. It should be pointed out that all the above-mentioned time delays are of the discrete nature. It is well known that dynamic systems usually have a spatial extent due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths, and...
there will be a distribution of propagation delays. In this case, the signal propagation is no longer instantaneous and cannot be modeled with discrete delays [23]. In [35], the robust $H_{\infty}$ filtering problem for uncertain singular systems with time-varying discrete and distributed delays has been investigated, and a sufficient condition for the existence of a filter has been expressed in terms of LMIs. The design problem of a delay-dependent robust $H_{\infty}$ controller has been studied for uncertain singular systems with time-varying discrete and distributed delays in [36]. Some results proposed in [36] have been further improved by Shu and Lam [37]. However, to the best of our knowledge, very little attention has been paid to discrete-time singular systems with discrete and distributed delays, which motivates the work of this paper.

In this paper, the problems of admissibility and dissipativity are investigated for discrete-time singular systems with discrete and distributed time-varying delays using the delay partitioning technique [38]. First, an admissibility criterion is established for the considered systems. Then, a sufficient condition is given to guarantee the dissipativity of the considered discrete-time singular systems with discrete and distributed delays, which motivates the work of this paper.

Notation: The notations used throughout this paper are fairly standard. $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the $n$-dimensional Euclidean space and the set of all $m \times n$ real matrices, respectively. The notation $X > Y (X \geq Y)$, where $X$ and $Y$ are symmetric matrices, means that $X - Y$ is positive definite (positive semidefinite). $I$ and $0$ represent the identity matrix and a zero matrix, respectively. The superscript "T" represents the transpose, and $\| \cdot \|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. For integers $a$ and $b$ with $a < b$, $\mathbb{N}(a, b) = \{a, a + 1, \ldots, b - 1, b\}$. $L_2[0, \infty)$ stands for the space of square summable infinite sequence on $[0, \infty)$. For an arbitrary matrix $B$ and two symmetric matrices $A$ and $C$:
\[
\begin{bmatrix}
A & B \\
0 & C
\end{bmatrix}
\]
denotes a symmetric matrix, where "$*$" denotes the term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Preliminaries

Consider the following discrete-time singular system with time-varying delays:

\[
\begin{aligned}
\dot{x}(k + 1) &= A_1 x(k) + A_2 x\left(k - d(k)\right) + A_3 \sum_{\nu=1}^{\tau(k)} x(k - \nu) + B_1 \omega(k), \\
x(k) &= C_1 x(k) + C_2 x\left(k - d(k)\right) + C_3 \sum_{\nu=1}^{\tau(k)} x(k - \nu) + D_1 \omega(k), \\
x(k) &= \phi(k),
\end{aligned}
\]

where $x(k) \in \mathbb{R}^n$ is the state vector, $z(k) \in \mathbb{R}^l$ is the output vector, and $\omega(k) \in \mathbb{R}^p$ is the disturbance input that belongs to $L_2[0, \infty)$. $\phi(k)$ is a compatible vector valued initial function. The matrix $E \in \mathbb{R}^{m \times n}$ may be singular and it is assumed that rank $E = r \leq n$. $A_1, A_2, B_1, C_1, C_2, C_3,$ and $D_1$ are known real constant matrices with appropriate dimensions. $d(k)$ and $\tau(k)$ denote discrete delay and distributed delay, respectively, and satisfy:

\[
0 < d_1 \leq d(k) \leq d_2, \quad 0 < \tau_1 \leq \tau(k) \leq \tau_2,
\]

where $d_1, d_2, \tau_1$, and $\tau_2$ are known integers.

Remark 1. In system (1), the term $\sum_{\nu=1}^{\tau(k)} x(k - \nu)$ originated from [39] is named as finite-distributed delay in the discrete-time setting, which can be regarded as the discrete analog of the finite-distributed delay $\int_{t-\tau(t)}^{t} x(s)ds$ in continuous-time systems. For the continuous-time case, the singular systems with finite-distributed delay has been studied extensively [35–37]. However, few results have been given for discrete-time singular systems with finite-distributed delay.

Throughout this paper, we shall use the following definitions and lemmas.

Definition 1 [331].

1. The discrete-time singular system:

\[
\begin{aligned}
\dot{x}(k + 1) &= A_1 x(k) + A_2 x\left(k - d(k)\right) + A_3 \sum_{\nu=1}^{\tau(k)} x(k - \nu) \\
x(k) &= \phi(k),
\end{aligned}
\]

is said to be regular and causal, if the pair $(E, A)$ is regular and causal.

2. The discrete-time singular system (3) is said to be stable, if its solution $x(k)$ satisfies:

\[
\lim_{k \to \infty} \|x(k)\| = 0.
\]

3. The discrete-time singular system (3) is said to be admissible, if it is regular, causal and stable.
We are now in a position to introduce the property of dissipativity.

Let the energy supply function of systems (1) be defined by

\[ G(\omega, z, \tau) = (z, Qz) + 2(z, S\omega)_T + \langle \omega, R\omega \rangle_T, \quad \forall \ \tau \geq 0, \]

(5)

where \( Q, S \) and \( R \) are real matrices with \( Q \) and \( R \) symmetric, \( \tau \) is an integer, and \( \langle a, b \rangle_T = \sum_{t=0}^{\tau} a(k)^T b(k) \).

Without loss of generality, it is assumed that \( \Omega \leq 0 \) and denoted that \( -\Omega = Q^T \Omega \). for some \( \Omega \).

**Definition 2.** Singular system (1) is said to be strictly \((Q, S, R)\)\(\gamma\)-dissipative if, for some scalar \( \gamma > 0 \), the following inequality:

\[ G(\omega, z, \tau) \geq \gamma \langle \omega, \omega \rangle_T, \quad \forall \ \tau \geq 0 \]

holds under zero initial condition.

**Lemma 1** [40]. For any matrix \( M > 0 \), integers \( \gamma_1 \) and \( \gamma_2 \) satisfying \( \gamma_2 > \gamma_1 \), and vector function \( \omega : \mathbb{N}[\gamma_1, \gamma_2] \rightarrow \mathbb{R}^n \), such that the sums concerned are well defined, then:

\[ (\gamma_2 - \gamma_1 + 1) \sum_{t=\gamma_1}^{\gamma_2} \omega(t)^T M \omega(t) \geq \sum_{t=\gamma_1}^{\gamma_2} \langle \omega(t)^T M \omega(t) \rangle. \]

(7)

**Lemma 2.** For any matrix \( \begin{bmatrix} M & S \\ * & M \end{bmatrix} \geq 0 \), integers \( d_1, d_2, d(k) \) satisfying \( d_1 \leq d(k) \leq d_2 \), and vector function \( x(k+\cdot) : \mathbb{N}[-d_2, -d_1] \rightarrow \mathbb{R}^n \), such that the sums concerned are well defined, then:

\[ -d_{12} \sum_{t=k-d_2}^{k-d_1} \zeta(t)^T M \zeta(t) \leq \sigma(k)^T \Omega \sigma(k). \]

(8)

where \( d_{12} = d_2 - d_1, \zeta(t) = x(t+1) - x(t) \) and

\[ \sigma(k) = \begin{bmatrix} x(k-d_1)^T \\ x(k-d_1)^T \\ x(k-d_2)^T \\ x(k-d_2)^T \end{bmatrix}, \]

\[ \Omega = \begin{bmatrix} M & -S \\ * & -2M + S^T & -S + M \\ * & * & -M \end{bmatrix}. \]

**Proof.** Denote:

\[ \eta_1(k) = \sum_{t=k-d_2}^{k-d_1-1} \zeta(t), \quad \eta_2(k) = \sum_{t=k-d_2}^{k-d_1-1} \zeta(t). \]

When \( d_1 < d(k) < d_2 \), according to Lemma 1, we have that:

\[ d_{12} \sum_{t=k-d_2}^{k-d_1} \zeta(t)^T M \zeta(t) = d_{12} \sum_{t=k-d_2}^{k-d_1-1} \zeta(t)^T M \zeta(t) + d_{12} \sum_{t=k-d_2}^{k-d_1-1} \zeta(t)^T M \zeta(t) \]

\[ \geq d_{12} \frac{d_2 - d_1}{d(k)^2} \eta_1(k)^T M \eta_1(k) + d_{12} \frac{d_2 - d_1}{d(k)^2} \eta_2(k)^T M \eta_2(k) \]

\[ = \eta_1(k)^T M \eta_1(k) + d_2 - d_1 \frac{d_2 - d_1}{d(k)^2} \eta_2(k)^T M \eta_2(k) + \frac{d_2 - d_1}{d(k)^2} \eta_2(k)^T M \eta_2(k). \]

(9)

On the other hand, it is clear that [11]:

\[ \begin{bmatrix} \sqrt{d_2 - d_1} \eta_1(k) \\ \sqrt{d_2 - d_1} \eta_2(k) \end{bmatrix}^T \begin{bmatrix} M & S \\ * & M \end{bmatrix} \begin{bmatrix} \sqrt{d_2 - d_1} \eta_1(k) \\ -\sqrt{d_2 - d_1} \eta_2(k) \end{bmatrix} \geq 0 \]

(10)

which implies:

\[ d_{12} \frac{d_2 - d_1}{d(k)^2} \eta_1(k)^T M \eta_1(k) + \frac{d_2 - d_1}{d(k)^2} \eta_2(k)^T M \eta_2(k) \geq \eta_1(k)^T S \eta_1(k) + \eta_2(k)^T S \eta_1(k). \]

(11)
Combining (9) and (11), we get that:

\[
d_{12} \sum_{s=k-d_2}^{k-d_1-1} \zeta(x)^{T} M \zeta(x) \geq \eta_1(k)^{T} M \eta_1(k) + \eta_2(k)^{T} M \eta_2(k) + \eta_1(k)^{T} S \eta_1(k) + \eta_2(k)^{T} S \eta_1(k) = \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix}^{T} \begin{bmatrix} M & S \\ S & M \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix}
\]

(12)

which implies (8) holds. Note that when \(d(k) = d_1\) or \(d(k) = d_2\), we have \(\eta_1(k) = 0\) or \(\eta_2(k) = 0\), respectively. Thus, (12) still holds. This completes the proof. □

Remark 2. It is noted that if \(S = 0\), the finite sum inequality (8) reduces the inequality (11) of [41]. Thus, inequality (8) in this paper is tighter than the existing one.

The first aim of this paper is to derive a delay-dependent condition, under which system (3) is admissible, and the second aim is to get a delay-dependent condition ensuring the admissibility and dissipativity of system (1).

Before giving our main results, for the sake’s of vector and matrix representation, the followings are denoted:

\[
Y(t) = \begin{bmatrix} x(k)^{T} \\ x(k - \frac{1}{m} d_1)^{T} \\ x(k - \frac{2}{m} d_1)^{T} \\ \vdots \\ x(k - \frac{n-1}{m} d_1)^{T} \end{bmatrix}^{T},
\]

\[
\eta(k) = \begin{bmatrix} Y(k)^{T} \\ x(k - d_1)^{T} \end{bmatrix}^{T},
\]

\[
\zeta(k) = \begin{bmatrix} \eta(k)^{T} x(k - d_2)^{T} x(k - d_3)^{T} \cdots \sum_{n=1}^{\frac{n_1}{m}} x(k - \nu)^{T} \omega(k)^{T} \end{bmatrix}^{T},
\]

\[
W_1 = [I_{mn} 0_{m \times n}],
\]

\[
W_2 = [0_{m \times n} I_{mn}],
\]

\[
g_i = \begin{bmatrix} 0_{n \times (l-1)n} & I_n & 0_{n \times (m-l+1)n} \end{bmatrix}, \quad l = 1, 2, \ldots, m + 1,
\]

and \(d_{12} = d_2 - d_1, \vartheta = \sum_{i=1}^{\frac{n_1}{m}+1} \frac{k}{m} \). This way, system (1) can be rewritten as:

\[
\begin{aligned}
&\dot{x}(k + 1) = A g_1 \eta(k) + A x(k - d(k)) + A_{\vartheta} \sum_{\nu=1}^{\frac{n_1}{m}} x(k - \nu) + B \omega(k), \\
x(k) = \phi(k), \quad k \in \mathbb{N}[\max(d_2, \tau_2), 0],
\end{aligned}
\]

(13)

and system (3) can be rewritten as:

\[
\begin{aligned}
&\dot{x}(k + 1) = A g_1 \eta(k) + A x(k - d(k)) + A_{\vartheta} \sum_{\nu=1}^{\frac{n_1}{m}} x(k - \nu), \\
x(k) = \phi(k), \quad k \in \mathbb{N}[\max(d_2, \tau_2), 0].
\end{aligned}
\]

(14)

3. Main results

In this section, we will investigate the problems of admissibility and dissipativity for discrete-time singular system with discrete and distributed time-varying delays.

Firstly, the following admissibility condition is given for system (3).

**Theorem 1.** Given an integer \(m > 0\), system (3) is admissible, if there exist matrices \(P, Q > 0, Z_1 > 0, Z_2 > 0, U > 0, S_i > 0 (i = 1, 2, \ldots, m), \sum_{i=1}^{m} S_i = 0, \) and \(W \) such that:

\[
\begin{aligned}
&\Xi_{11} \Xi_{12} \Xi_{13} \Xi_{14} \Xi_{15} g_i^{T} A_{i}^{T} P \\
&* \Xi_{22} \Xi_{23} \quad 0 \quad A_{i}^{T}D \quad A_{i}^{T} P \\
&* \quad * \quad 0 \quad 0 \\
&* \quad \quad * \quad -U \quad A_{i}^{T}D \quad A_{i}^{T} P \\
&* \quad * \quad \quad * \quad -D \quad 0 \\
&* \quad * \quad * \quad * \quad -P
\end{aligned}
\]

\[
< 0,
\]

(15)

where \(R \in \mathcal{R}^{n \times (n-r)}\) is any matrix with full column and satisfies \(E^{T} R = 0\), and \(D = \left(\frac{d}{m}\right) \sum_{i=1}^{m} S_i + d_{12}^{2} S_{m+1} \) and
Lyapunov functional for system (3):

$$Z_{11} = -g_1^TPEg_1 + W_1^TQW_1 - W_2^TQW_2 - \sum_{i=1}^{m} (g_i - g_i+1)^T E_S E (g_i - g_i+1) + g_{m+1}^T Z_{m+1} + (d_{12} + 1) g_{m+1}^T Z_{m+1}$$

$$+ \partial g_{m+1}^T U g_1 - g_{m+1}^T E S_{m+1} E g_{m+1} + g_1^T W R A g_1 + g_1^T A^T R W E g_1.$$ 

$$Z_{12} = g_{m+1}^T E S_{m+1} E - g_{m+1}^T E^T \gamma E + g_1^T W R A d,$$

$$Z_{13} = g_{m+1}^T E^T \gamma E,$$

$$Z_{14} = g_1^T W R A d,$$

$$Z_{15} = g_1^T (A - E)^T D,$$

$$Z_{22} = -Z_2 - 2E^T S_{m+1} E + E^T \gamma E + E^T \gamma^T E,$$

$$Z_{23} = -E^T \gamma E + E^T S_{m+1} E,$$

$$Z_{33} = -Z_1 - E^T S_{m+1} E.$$

**Proof.** Under the given condition, we first show that system (3) is regular and causal. Since \( \text{rank} E = r \), we choose two nonsingular matrices \( M \) and \( G \) such that:

$$MEG = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \quad (16)$$

Set:

$$MAG = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad G^T W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad M^{-T} R = \begin{bmatrix} 0 \\ I \end{bmatrix} F, \quad (17)$$

where \( F \in \mathbb{R}^{(r - f)\times(n - t)} \) is any nonsingular matrix. It can be seen that \( Z_{11} < 0 \) implies:

$$-E^T PE + WR^A A + A^T RW^T - E^T S_1 E < 0. \quad (18)$$

Pre-multiplying and post-multiplying (18) by \( G^T \) and \( G \), respectively, we have \( W_2 F^T A_4 + A_4^T FS_1^T < 0 \), which implies \( A_4 \) is nonsingular. Thus, the pair \((E, A)\) is regular and causal. According to Definition 1, system (3) is regular and causal.

Next we will show that system (3) is stable. To the end, we define \( \delta(k) = x(k + 1) - x(k) \) and consider the following Lyapunov functional for system (3):

$$V(x(k), k) = \sum_{i=1}^{5} V_i(x(k), k), \quad (19)$$

where

$$V_1(x(k), k) = x(k)^T E^T P E x(k),$$

$$V_2(x(k), k) = \sum_{s-k+d_1}^{k-1} \gamma(s)^T Q \gamma(s),$$

$$V_3(x(k), k) = \sum_{s-k-d_2}^{k-d_2} x(s)^T Z_1 x(s) + \sum_{j-d_2+1}^{k-d_1} \sum_{s-k+1-j}^{k-d_1} x(s)^T Z_2 x(s),$$

$$V_4(x(k), k) = \sum_{d_{12}}^{d_{12} + 1} \sum_{s-k+g}^{k-1} \delta(s)^T E_S E \delta(s) + \frac{d_{12}}{m} \sum_{s-k+g}^{k-1} \sum_{s-k}^{1} \delta(s)^T E S_{m+1} E \delta(s),$$

$$V_5(x(k), k) = \tau_2 \sum_{d_{13}}^{d_{13} + 1} \sum_{p-1}^{d_{13}} \sum_{s-k-p}^{k-1} x(z)^T U x(z).$$

Then, along the trajectory of system (3), we have that:
\[ \Delta V_1(k) = x(k+1)^T E^T P E x(k+1) - x(k)^T E^T P E x(k) - \eta(k) g^T_1 E^g_1 \eta(k), \]  
(20)

\[ \Delta V_2(k) = \Upsilon(k)^T Q \Upsilon(k) - \Upsilon \left( k - \frac{d_1}{m} \right)^T Q \Upsilon \left( k - \frac{d_1}{m} \right) = \eta(k)^T W_1 Q W_1 \eta(k) - \eta(k)^T W_2 Q W_2 \eta(k), \]  
(21)

\[ \Delta V_3(k) = x(k-d_1)^T Z_1 x(k-d_1) - x(k-d_2)^T Z_1 x(k-d_2) + (d_{12} + 1) x(k-d_1)^T Z_2 x(k-d_1) - \sum_{s=k-d_2}^{k-d_1} \chi(s)^T Z_2 \chi(s) \leq \eta(k)^T G^t_1 Z_1 g_{m+1} \eta(k) - x(k-d_1)^T Z_1 x(k-d_1) + (d_{12} + 1) \eta(k)^T G^t_1 Z_2 g_{m+1} \eta(k) - x(k-d_2)^T Z_2 x(k-d_2). \]  
(22)

\[ \Delta V_4(k) = \left( \frac{d_1}{m} \right)^2 \sum_{i=1}^{m k} \delta(s)^T E^T S E \delta(s) - \sum_{i=1}^{m k} \left( \frac{d_1}{m} \right)^2 \sum_{s=k-d_2}^{k-d_1} \delta(s)^T E^T S E \delta(s) \]  

\[ -d_{12} \sum_{s=k-d_2}^{k-d_1} \delta(s)^T E^T S m+1 E \delta(s). \]  
(23)

\[ \Delta V_5(k) = \partial x(k)^T U x(k) - \tau_2 \sum_{i=1}^{n_k} x(k-v)^T U x(k-v) \leq \partial \eta(k)^T g^1 U g_1 \eta(k) - \tau_2 \sum_{i=1}^{\tau_k} x(k-v)^T U x(k-v). \]  
(24)

By use of Lemma 2, we get:

\[ -d_{12} \sum_{s=k-d_2}^{k-d_1} \delta(s)^T E^T S m+1 E \delta(s) \leq \begin{bmatrix} \eta(k) \\ x(k-d(k)) \end{bmatrix}^T \Gamma \begin{bmatrix} \eta(k) \\ x(k-d(k)) \end{bmatrix}, \]  
(25)

where

\[ \Gamma = \begin{bmatrix} -g^T_{m+1} E^T S_{m+1} g_{m+1} & g^T_{m+1} E^T S_{m+1} \gamma \eta + g^T_{m+1} E^T \gamma \eta + g^T_{m+1} E^T \gamma E -2E^T S_{m+1} E + E^T \gamma E + E^T \gamma E -E^T \gamma E + E^T S_{m+1} E + E^T S_{m+1} E \end{bmatrix}. \]

Furthermore, applying Lemma 1, we obtain:

\[ -\frac{d_1}{m} \sum_{i=1}^{m k} \sum_{s=k-d_2}^{k-d_1} \delta(s)^T E^T S E \delta(s) \leq -\sum_{i=1}^{m k} \sum_{s=k-d_2}^{k-d_1} \sum_{s=k-d_2}^{k-d_1} \delta(s)^T E^T S E \delta(s) \]  

\[ = -\sum_{i=1}^{m k} \eta(k)^T (g_i - g_{i+1})^T E^T S E (g_i - g_{i+1}) \eta(k), \]  
(26)

and

\[ -\tau_2 \sum_{i=1}^{n_k} x(k-v)^T U x(k-v) \leq -\sum_{i=1}^{n_k} x(k-v)^T U x(k-v). \]  
(27)

On the other hand, it is clear:

\[ f(k) = 2x(k)^T W R^T E x(k+1) - 2\eta(k)^T g^1 E^T W R^T E x(k+1) = 0. \]  
(28)

Thus,

\[ \Delta V(k) = \sum_{i=1}^{s} \Delta V_i(k) + f(k) \leq \tilde{\xi}(k)^T \Theta \tilde{\xi}(k), \]  
(29)

where

\[ \tilde{\xi}(k) = \begin{bmatrix} \eta(k)^T & x(k-d(k))^T & x(k-d_2)^T \sum_{i=1}^{\tau_k} x(k-v)^T \end{bmatrix}^T, \]

\[ \Theta = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} \\ * & \Xi_{22} & \Xi_{23} & 0 \\ * & * & \Xi_{33} & 0 \\ * & * & * & -\Upsilon \end{bmatrix} + \begin{bmatrix} g^T_1 A^T_g & g^T_1 A^T_d & g^T_1 (A-E)^T & g^T_1 (A-E)^T \end{bmatrix} D \begin{bmatrix} A^T_g & A^T_d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

According to Schur complement, we get from (15) that \( \Theta < 0. \) Therefore,
\[ 
\Delta V(k) \leq -\alpha \|x(k)\|^2, \quad (30) 
\]
where \( \alpha = -\lambda_{\text{max}}(\Theta) > 0 \). Thus, we have:

\[
\sum_{i=0}^{k} \|x(i)\|^2 \leq \frac{1}{\alpha} V(0) < \infty 
\]

(31)

that is, series \( \sum_{i=0}^{\infty} \|x(i)\|^2 \) converge, which implies that (4) holds. According to **Definition 1**, system (3) is stable. This completes the proof. \( \Box \)

**Remark 3.** In terms of the delay partitioning technique, **Theorem 1** provides an admissibility condition for discrete-time singular systems with mixed time-varying delays. It is noted that the conservatism of the given condition is reduced as the number of delay partitioning grows.

Specifically, when there is no distributed delay, system (3) reduces to:

\[
\begin{align*}
E \{x(k + 1) = A x(k) + A d x(k - d(k)), \\
x(k) = \phi(k), \quad k \in \mathbb{N} \cup \{-\max\{d_2, T_2\}, 0\}.
\end{align*}
\]

(32)

Then we can get the following corollary based on **Theorem 1**.

**Corollary 1.** Given an integer \( m > 0 \), system (32) is admissible, if there exist matrices \( P, Q > 0, Z_1 > 0, Z_2 > 0, S_i > 0 (i = 1, 2, \ldots, m + 1) \), and \( W \) such that:

\[
\begin{bmatrix}
\bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} & \bar{Z}_{15} & g_1^T A_1 P \\
* & \bar{Z}_{22} & \bar{Z}_{23} & A_1^T D & A_1^T P \\
* & * & \bar{Z}_{33} & -D & 0 \\
* & * & * & -D & 0 \\
* & * & * & * & -P
\end{bmatrix} < 0,
\]

(33)

where \( \bar{Z}_{12}, \bar{Z}_{13}, \bar{Z}_{15}, \bar{Z}_{22}, \bar{Z}_{23}, \bar{Z}_{33} \), and \( D \) follow the same definitions as those in **Theorem 1**, and

\[
\begin{align*}
\bar{Z}_{11} &= -g_1^T E_1 P g_1 + W_1^T Q W_1 - W_1^T W_1 - \sum_{i=1}^{m} (g_i - g_{i+1})^T E_i S_i E_i(g_i - g_{i+1}) + g_{m+1}^T D_{m+1} Z_1 g_{m+1} + (d_1 + 1) g_1^T D_{m+1} Z_2 g_{m+1} \\
&- g_{m+1}^T E_{m+1} S_{m+1} E_{m+1} + g_1^T W R^T A g_1 + g_1^T A^T R W^T g_1.
\end{align*}
\]

The following theorem presents the result on the dissipativity analysis for system (1) based on **Theorem 1**.

**Theorem 2.** Given an integer \( m > 0 \), system (1) is admissible and strictly \((Q, S, R) - \gamma\)-dissipative, if there exist matrices \( P, Q > 0, Z_1 > 0, Z_2 > 0, U > 0, S_i > 0 (i = 1, 2, \ldots, m) \), \( \bar{S}_{m+1} \), and \( W \), and a scalar \( \gamma > 0 \) such that:

\[
\begin{bmatrix}
\bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} & \bar{Z}_{14} & A_1 & \bar{Z}_{15} & g_1^T A_1 P & g_1^T C_1 Q^T \\
* & \bar{Z}_{22} & \bar{Z}_{23} & 0 & -C_1^T S & A_1^T D & A_1^T P & C_1^T Q^T \\
* & * & \bar{Z}_{33} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -U & -C_1^T S & A_1^T D & A_1^T P & C_1^T Q^T \\
* & * & * & * & A_2 & B_1^T D & B_2^T P & D_{10}^T Q^T \\
* & * & * & * & * & -P & 0 & 0 \\
* & * & * & * & * & * & -I
\end{bmatrix} < 0,
\]

(34)

where \( \bar{Z}_{11}, \bar{Z}_{12}, \bar{Z}_{13}, \bar{Z}_{14}, \bar{Z}_{15}, \bar{Z}_{22}, \bar{Z}_{23}, \bar{Z}_{33} \), and \( D \) follow the same definitions as those in **Theorem 1**, and

\[
A_1 = g_1^T W R^T B_1 - g_1^T C_1 S,
\]

\[
A_2 = -R + \gamma I - S^T D_{10} - D_{10}^T S.
\]

**Proof.** It is clear that (34) implies (15). Therefore, system (1) with \( \omega(t) = 0 \) is stable according to **Theorem 1**. To prove the dissipativity of system (1), we consider Lyapunov functional (19) and the following index for system (1):

\[
J_{x_o} = \sum_{k=0}^{N} \|x(k)\|^2 + 2z(k)^T (Qz(k) + \omega(k)^T (R - \gamma I) \omega(k)).
\]

(35)
Applying a similar analysis method employed in the proof of Theorem 1, we have that:
\[
\sum_{k=0}^{r} \Delta V(k) - J_{x,0} \leq \sum_{k=0}^{r} \xi(k)^T \Xi(k),
\]
where
\[
\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Delta_1 \\
* & \Xi_{22} & \Xi_{23} & -C_d^T S \\
* & * & \Xi_{33} & 0 \\
* & * & * & \Delta_2
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & \mathcal{D} & 0 & -0 & Q & 0 & 0
\end{bmatrix}.
\]
According to Schur complement, we get from (34) that $\Xi < 0$. Therefore:
\[
\sum_{k=0}^{r} \Delta V(k) \leq J_{x,0}
\]
which implies:
\[
V(x(\tau+1)) - V(x(0)) \leq J_{x,0}.
\]
Thus, (6) holds under zero initial condition. Therefore, according to Definition 2, systems (1) is strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative. This completes the proof. □

Remark 4. Theorem 2 provides a delay-dependent condition on the dissipativity of singular system (1). It should be pointed out that by setting $\delta = -\gamma$ and minimizing $\delta$ subject to (34), we can obtain the optimal dissipativity performance $\gamma^*$ (by $\gamma^* = -\delta$).

Similarly, we can get the following corollary on the dissipativity for the following singular system without distributed delay:
\[
\begin{align*}
\dot{x}(k+1) &= A x(k) + A_d x(k - d(k)) + B_d w(k), \\
\xi(k) &= C x(k) + C_d x(k - d(k)) + D_d w(k), \\
x(k) &= \phi(k), \quad k \in \mathbb{N}[-\max(d_2, \tau_2), 0].
\end{align*}
\]

Corollary 2. Given an integer $m > 0$, system (39) is admissible and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\gamma$-dissipative, if there exist matrices $P, Q > 0, Z_1 > 0, Z_2 > 0, S_i > 0$ ($i = 1, 2, \ldots, m$), $X_m$, $S_m$, and $W$, and a scalar $\gamma > 0$ such that:
\[
\begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Delta_1 & \Xi_{15} & g[A]^T P & g[C]^T Q^T \\
* & \Xi_{22} & \Xi_{23} & -C_d^T S & A_d^T D & A_d^T P & C_d^T Q^T \\
* & * & \Xi_{33} & 0 & 0 & 0 & 0 \\
* & * & * & \Delta_2 & B_d^T D & B_d^T P & D_d^T Q^T < 0
\end{bmatrix}
\]
where $\Xi_{11}$ follows the same definition as that in Corollary 1, $\Xi_{12}, \Xi_{13}, \Xi_{15}, \Xi_{22}, \Xi_{23}, \Xi_{33}$, and $\mathcal{D}$ follow the same definitions as those in Theorem 1, and $\Delta_1$ and $\Delta_2$ follow the same definitions as those in Theorem 2.

4. Numerical examples

In this section, three numerical examples are introduced to demonstrate the effectiveness of the proposed methods.

Example 1. Consider system (32) with
\[
E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}.
\]
In this example, we choose $d_1 = 3$ and
Applying the methods of [32,33] and Corollary 1 of this paper, the allowable maximum values of \(d_2\) ensuring the admissibility of the considered system are listed in Table 1, which shows our condition gives better results than those in [32,33] even for the case of \(m = 1\).

**Example 2.** Consider system (3) with

\[
E = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} 5.36 & 9.88 \\ 6.68 & 6.94 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1.50 & -1.55 \\ -1.50 & -1.15 \end{bmatrix}, \quad A_r = \begin{bmatrix} 3.68 & 2.00 \\ 1.84 & 1.00 \end{bmatrix}.
\]

In this example, we choose \(d_1 = 6, \quad d_2 = 14, \quad \tau_1 = 5, \quad \tau_2 = 12, \) and

\[
R = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.
\]

By using the Matlab LMI toolbox to solve the LMI (15) with \(m = 2\), it can be checked that considered system is admissible.

**Example 3.** Consider system (1) with

\[
E = \begin{bmatrix} 56 & 35 \\ 64 & 40 \end{bmatrix}, \quad A = \begin{bmatrix} 132.48 & 90 \\ 182.72 & 125 \end{bmatrix},
\]

\[
A_d = \begin{bmatrix} -24.4 & -16.05 \\ -33.6 & -22.2 \end{bmatrix}, \quad A_r = \begin{bmatrix} 30.72 & 19.36 \\ 46.08 & 29.04 \end{bmatrix},
\]

\[
C_r = \begin{bmatrix} 0.03 & 0.01 \\ 0.04 & 0.02 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0.4 & 0.2 \\ 6 & 4 \end{bmatrix}, \quad C_s = \begin{bmatrix} 1.8 & 1.3 \\ 0 & 0 \end{bmatrix},
\]

\[
B_w = \begin{bmatrix} 0.2 & 1.4 \\ 0.1 & 0.2 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0.1 & 0.01 \\ 0.2 & 0 \end{bmatrix}.
\]

In this example, we choose \(d_1 = 6, \quad d_2 = 9, \quad \tau_1 = 3, \quad \tau_2 = 8, \) and

\[
Q = \begin{bmatrix} -0.04 & 0 \\ 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0.6 \\ 1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

The purpose of this example is to find the optimal dissipativity performance \(\gamma\) such that the considered system is admissible and strictly \((Q,S,R)\)-\(\gamma\)-dissipative. By application of Theorem 2 with \(R = [8 - 7]^T\), the optimal dissipativity performance \(\gamma\) for different partitioning size is given in Table 2, from which we can find that the larger partition number \(m\) corresponds to the larger dissipativity performance \(\gamma\).

### Table 2

<table>
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<th>(m)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>6</th>
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<td>0.3705</td>
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### 5. Conclusions

The problems of admissibility and dissipativity have been studied for discrete-time singular systems with mixed delays. The mixed time delays consist of both discrete and distributed delays. By taking advantage of the delay partitioning technique, a delay-dependent criterion has been given to ensure singular systems admissible. Based on the criterion, a
delay-dependent dissipativity condition has derived. All the results presented depend upon not only discrete delay and distributed delay, but also rely upon the partitioning size. Three numerical examples have been proposed to demonstrate the effectiveness of the proposed methods.

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References