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New approaches on stability criteria for neural networks with interval time-varying delays

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ABSTRACT

This paper concerns the problem of delay-dependent stability criteria for neural networks with interval time-varying delays. First, by constructing a newly augmented Lyapunov–Krasovskii functional and combining with a reciprocally convex combination technique, less conservative stability criterion is established in terms of linear matrix inequalities (LMIs), which will be introduced in Theorem 1. Second, by taking different interval of integral terms of Lyapunov–Krasovskii functional utilized in Theorem 1, further improved stability criterion is proposed in Theorem 2. Third, a novel approach which divides the bounding of activation function into two subinterval are proposed in Theorem 3 to reduce the conservatism of stability criterion. Finally, through two well-known numerical examples used in other literature, it will be shown the proposed stability criteria achieves the improvements over the existing ones and the effectiveness of the proposed idea.

1. Introduction

Recently, neural networks have been found successful applications in various fields including image and signal processing, pattern recognition, fault diagnosis, associative memories, fixed-point computations, combinatorial optimization, and other scientific areas (for instance, see [1–4]). Since these applications heavily depend on the dynamic behavior of the equilibrium point, the stability analysis of the equilibrium points of the designed network has been one of important issue. In the implementation of neural networks, time delays frequently occur due to the finite switching speed of amplifies and may cause instability or oscillation of neural networks. Therefore, considerable efforts have been done to asymptotic stability analysis of neural networks with time-delays [5–36]. Especially, delay-dependent stability analysis has been investigated by many researchers [18–36] since it is well known that delay-dependent stability criteria are generally less conservative than delay-independent ones when the sizes of time-delays are small.

In the field of delay-dependent stability analysis, an important index for checking the conservatism is to find maximum delay bounds such that the asymptotic stability of the designed network can be guaranteed for any admissible delays less than maximum delay bounds. Thus, how to choose Lyapunov–Krasovskii functional and obtain an upper bound of time-derivative of it play key roles to increase the feasible region of stability criteria.

Since Li et al. [21] pointed that the stability conditions are hardly improved by using the same Lyapunov–Krasovskii functional, delay-partitioning approach, which was firstly introduced by Gu [38], has been attracted by many researchers. One of the

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main advantage of this approach is to estimate more tighter upper bounds than the the results without delay-partitioning approach. In this regard, with the idea of delay-partitioning, new exponential stability criterion for neural networks with constant time-delay was investigated in [29]. Hu, Gao and Zheng [30] investigated the asymptotic stability for a class of cellular neural networks with interval time-varying delays by introducing a novel Lyapunov functional which divide the lower bound of the time-varying delay. The stability analysis using delay-partitioning was further investigated for stochastic Hopfield neural networks with constant time-delays and norm-bounded parameter uncertainties [31]. In [32], a piecewise delay method which divides delay interval into two was proposed to reduce the conservatism of delay-dependent stability criteria for neural networks with interval time-varying delays. Recently, the idea of the proposed method [30] was extended to the problem of exponential stability on stochastic neural networks with discrete interval and distributed delays [33]. Very recently, by employing the improved delay-partitioning technique and general convex combination, delay-derivative-dependent stability criteria for delayed neural networks with unbounded distributed delay were proposed in [35] and further improvement results on [35] are in [36]. All of the works [29–36] mentioned focused on the application of delay-partitioning approach. However, as mentioned in [37], the choice of activation functions influence the ability and performance of neural networks. Thus, the bounding properties of activation function is another important factor to determine the feasible region of stability criteria.

Motivated by this mentioned above,in this paper, three new delay-dependent stability criteria for neural networks with interval time-varying delays will be proposed by employing different approaches. The contribution of this paper lies in three aspects.

1. Unlike the method of [29–36], no delay-partitioning methods are utilized. Instead, by taking more information of states and activation functions as augmented vectors, an augmented Lyapunov–Krasovskii's functional is proposed. Then, inspired by the work of [39–42], a sufficient condition such that the considered neural networks are asymptotically stable is derived in terms of linear matrix inequalities (LMIs) which will be presented in Theorem 1 and 2.
2. A novel approach partitioning the bounding of activation function will be proposed for the first time. As a tradeoff between time-consuming and improvement of the feasible region, the bounding of activation function is divided into two subintervals.

Through two well-known numerical examples, it will be shown that in spite of no employing delay-partitioning approaches the proposed stability criteria can provide larger delay bounds than the recent results in which delay-partitioning techniques were utilized.*Notation*. Throughout this brief, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices. For symmetric matrices X and Y, the notation $X>Y$ (respectively, $X\geqslant Y$) means that the matrix $X-Y$ is positive definite, (respectively, nonnegative). $diag\{\cdots\}$ denotes the block diagonal matrix. ☆ represents the elements below the main diagonal of a symmetric matrix. The subscript 'T' denotes the transpose of the matrix.

2. Problem statement

Consider the following neural networks with interval time-varying delays:

$$\dot{y}(t) = -Ay(t) + W_0 g(y(t)) + W_1 g(y(t-h(t))) + b, \tag{1}$$

where $y(t)=[y_1(t),\ldots,y_n(t)]^T\in\mathbb{R}^n$ is the neuron state vector, n denotes the number of neurons in a neural network, $g(y(t))=[g_1(y_1(t)),\ldots,g_n(y_n(t))]^T\in\mathbb{R}^n$ means the neuron activation function, $g(y(t-h(t)))=[g_1(y_1(t-h(t))),\ldots,g_n(y_n(t-h(t)))]^T\in\mathbb{R}^n$, $A=diag\{a_i\}\in\mathbb{R}^{n\times n}$ is a positive diagonal matrix, $W_0=(w^0_{ij})_{n\times n}\in\mathbb{R}^{n\times n}$ and $W_1=(w^1_{ij})_{n\times n}\in\mathbb{R}^{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons, and $b=[b_1,b_2,\ldots,b_n]^T\in\mathbb{R}^n$ represents a constant input vector.

The delay, $h(t)$, is a time-varying continuous function satisfying

$$0\leqslant h_L\leqslant h(t)\leqslant h_U, \quad \dot{h}(t)\leqslant h_D, \tag{2}$$

where h_L and h_U are positive scalars and h_D is any constant one.

The activation functions, $g_i(y_i(t)),\ i=1,\ldots,n$, are assumed to bounded and hold the following condition:

$$\begin{aligned} &k^-\leqslant\frac{g_i(u)-g_i(v)}{u-v}\leqslant k_i^+, u,v\in\mathbb{R},\\ &u\neq v,\ i=1,\ldots,n, \end{aligned} \tag{3}$$

where k_i^- and k_i^+ are constant values.

For simplicity, in stability analysis of the neural networks (1), the equilibrium point $y^*=[y_1^*,\ldots,y_n^*]^T$ whose uniqueness has been reported in [18] is shifted to the origin by utilizing the transformation $x(\cdot)=y(\cdot)-y^*$, which leads the system (1) to the following form:

$$\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 f(x(t-h(t))), \tag{4}$$

where $x(t)=[x_1(t),\ldots,x_n(t)]^T\in\mathbb{R}^n$ is the state vector of the transformed system, $f(x(t))=[f_1(x_1(t)),\ldots,f_n(x_n(t))]^T$ and $f_j(x_j(t))=g_j(x_j(t)+y_j^*)-g_j(y_j^*)$ with $f_j(0)=0 (j=1,\ldots,n)$.

It should be noted that the activation functions $f_i(\cdot)(i = 1, \ldots, n)$ satisfy the following condition:

$$k_i^- \leqslant \frac{f_i(u) - f_i(v)}{u - v} \leqslant k_i^+, u, v \in \mathbb{R}, \quad u \neq v, \; i = 1, \ldots, n. \tag{5}$$

If $v = 0$ in (5), then we have

$$k_i^- \leqslant \frac{f_i(u)}{u} \leqslant k_i^+, \quad \forall u \neq 0, \; i = 1, \ldots, n, \tag{6}$$

which is equivalent to

$$[f_i(u) - k_i^- u][f_i(u) - k_i^+ u] \leqslant 0, \quad i = 1, \ldots, n. \tag{7}$$

The objective of this paper is to investigate the delay-dependent stability conditions of system (4) which will be conducted in Section 3.

Before deriving our main results, we state the following lemmas.

Lemma 1. *For any constant positive-definite matrix $M \in \mathbb{R}^{n\times n}$ and $\beta \leqslant s \leqslant \alpha$, the following inequalities hold:*

$$(\alpha - \beta)\int_\beta^\alpha \dot{x}^T(s)M\dot{x}(s)ds \geqslant \left(\int_\beta^\alpha \dot{x}(s)ds\right)^T M\left(\int_\beta^\alpha \dot{x}(s)ds\right), \tag{8}$$

$$\frac{(\alpha - \beta)^2}{2}\int_\beta^\alpha \int_s^\alpha \dot{x}^T(u)M\dot{x}(u)duds \geqslant \left(\int_\beta^\alpha \int_s^\alpha \dot{x}(u)duds\right)^T M\left(\int_\beta^\alpha \int_s^\alpha \dot{x}(u)uds\right). \tag{9}$$

Proof. According to Jensen's inequality in [43], one can obtain (8). Moreover, the following inequality holds

$$(\alpha - s)\int_s^\alpha \dot{x}^T(u)M\dot{x}(u)du \geqslant \left(\int_s^\alpha \dot{x}(u)du\right)^T M\left(\int_s^\alpha \dot{x}(u)du\right). \tag{10}$$

By Schur Complements [44], Eq. (10) is equivalent to the following

$$\begin{bmatrix} \int_s^\alpha \dot{x}^T(u)M\dot{x}(u)du & \int_s^\alpha \dot{x}^T(u)du \\ \int_s^\alpha \dot{x}(u)du & (\alpha - s)M^{-1} \end{bmatrix} \geqslant 0. \tag{11}$$

Integration of (11) from β to α yields

$$\begin{bmatrix} \int_\beta^\alpha \int_s^\alpha \dot{x}^T(u)M\dot{x}(u)duds & \int_\beta^\alpha \int_s^\alpha \dot{x}^T(u)duds \\ \int_\beta^\alpha \int_s^\alpha \dot{x}(u)duds & \int_\beta^\alpha (\alpha - s)M^{-1}ds \end{bmatrix} \geqslant 0. \tag{12}$$

Therefore, the inequality (12) is equivalent to the inequality (9) according to Schur Complements. This complete the proof. □

Lemma 2 [45]. *Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that rank$(B) < n$. Then, the following statements are equivalent:*

(1) $\zeta^T\Phi\zeta < 0, \; B\zeta = 0, \; \zeta \neq 0,$
(2) $(B^\perp)^T\Phi B^\perp < 0$, *where $B^\perp$ is a right orthogonal complement of B.*

3. Main results

In this section, using augmented Lyapunov–Krasovskii functionals and some new approaches, novel delay-dependent stability criteria for systems (4) will be proposed. For the sake of simplicity of matrix representation, $e_i(i = 1, \ldots, 17) \in \mathbb{R}^{17n\times n}$ are defined as block entry matrices. (For example, $e_3^T = [0 0 I 0 0 0 0 0 0 0 0 0 0 0 0 0 0]$). The notations for some matrices are defined as:

$$\zeta(t) = \Big[x^T(t)\; x^T(t - h(t))\; x^T(t - h_L)\; x^T(t - h_U)\; \dot{x}^T(t)\; \dot{x}^T(t - h_L)\; \dot{x}^T(t - h_U)\; \int_{t-h_L}^t x^T(s)ds\; \int_{t-h(t)}^{t-h_L} x^T(s)ds\; \int_{t-h_U}^{t-h(t)} x^T(s)ds$$
$$\times f^T(x(t))\; f^T(x(t - h(t)))\; f^T(x(t - h_L))\; f^T(x(t - h_U))\; \int_{t-h_L}^t f^T(x(s))ds\; \int_{t-h(t)}^{t-h_L} f^T(x(s))ds\; \int_{t-h_U}^{t-h(t)} f^T(x(s))ds\Big],$$

$$\alpha(t) = \begin{bmatrix} x^T(t) & \dot{x}^T(t) & f^T(x(t)) \end{bmatrix}, \quad \beta(t) = \begin{bmatrix} x^T(t) & f^T(x(t)) \end{bmatrix},$$

$$\Pi_1 = [e_1 \quad e_3 \quad e_4 \quad e_8 \quad e_9 + e_{10} \quad e_{15} \quad e_{16} + e_{17}],$$

$$\Pi_2 = [e_5 \quad e_6 \quad e_7 \quad e_1 - e_3 \quad e_3 - e_4 \quad e_{11} - e_{13} \quad e_{13} - e_{14}],$$

$$\Pi_3 = [e_1 \quad e_5 \quad e_{11}], \quad \Pi_4 = [e_3 \quad e_6 \quad e_{13}], \quad \Pi_5 = [e_4 \quad e_7 \quad e_{14}],$$

$$\Pi_6 = [e_1 \quad e_{11}], \quad \Pi_7 = [e_2 \quad e_{12}], \quad \Pi_8 = [e_8 \quad e_1 - e_3 \quad e_{15}],$$

$$\Pi_9 = [e_9 \quad e_3 - e_2 \quad e_{16} \quad e_{10} \quad e_2 - e_4 \quad e_{17}],$$

$$\Gamma = [-A \quad 0 \quad 0 \quad 0 \quad -I \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad W_0 \quad W_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$\Phi = e_{11}\Lambda e_5^T + e_5\Lambda e_{11}^T + e_1K_p\Delta e_5^T + e_5\Delta K_p e_1^T - e_{11}\Delta e_5^T - e_5\Delta e_{11}^T,$$

$$\Xi = (h_L^2/2)^2 e_5Q_3e_5^T - (h_Le_1 - e_8)Q_3(h_Le_1 - e_8)^T + \left((h_U^2 - h_L^2)/2\right)^2 e_5Q_4e_5^T - ((h_U - h_L)e_1 - e_9 - e_{10})Q_4((h_U - h_L)e_1 - e_9 - e_{10})^T,$$

$$\Psi = h_Le_1Q_5e_1^T + h_Le_5Q_6e_5^T + (h_U - h_L)e_1Q_7e_1^T + (h_U - h_L)e_5Q_8e_5^T + e_1P_1e_1^T + e_3(-P_1 + P_2)e_3^T + e_2(-P_2 + P_3)e_2^T - e_4P_3e_4^T,$$

$$\begin{aligned} \Upsilon &= e_1(K_p + K_m)H_1e_{11}^T + e_{11}H_1(K_p + K_m)e_1^T - 2e_{11}H_1e_{11}^T - 2e_1K_mH_1K_pe_1^T + e_2(K_p + K_m)H_2e_{12}^T + e_{12}H_2(K_p + K_m)e_2^T \\ &\quad - 2e_{12}H_2e_{12}^T - 2e_2K_mH_2K_pe_2^T + e_3(K_p + K_m)H_3e_{13}^T + e_{13}H_3(K_p + K_m)e_3^T - 2e_{13}H_3e_{13}^T - 2e_3K_mH_3K_pe_3^T \\ &\quad + e_4(K_p + K_m)H_4e_{14}^T + e_{14}H_4(K_p + K_m)e_4^T - 2e_{14}H_4e_{14}^T - 2e_4K_mH_4K_pe_4^T, \end{aligned}$$

$$\begin{aligned} \Sigma_1 &= \Xi + \Phi + \Psi + \Pi_1\mathcal{R}\Pi_2^T + \Pi_2\mathcal{R}\Pi_1^T + \Pi_3\mathcal{N}\Pi_3^T + \Pi_4(-\mathcal{N} + \mathcal{M})\Pi_4^T - \Pi_5\mathcal{M}\Pi_5^T + \Pi_6\mathcal{G}\Pi_6^T - (1 - h_D)\Pi_7\mathcal{G}\Pi_7^T \\ &\quad + h_L^2\Pi_3\mathcal{Q}_1\Pi_3^T - \Pi_8\mathcal{Q}_1\Pi_8^T + (h_U - h_L)^2\Pi_3\mathcal{Q}_2\Pi_3^T - \Pi_9\begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix}\Pi_9^T. \end{aligned} \tag{13}$$

Now, we have the following theorem.

Theorem 1. *For given positive scalars h_L and h_U, any scalar h_D, diagonal matrices $K_m = diag\{k_1^-, \ldots, k_n^-\}$ and $K_p = diag\{k_1^+, \ldots, k_n^+\}$, the system (1) is asymptotically stable for $0 \leqslant h_L \leqslant h(t) \leqslant h_U$ and $\dot{h}(t) \leqslant h_D$ if there exist positive diagonal matrices $\Lambda = diag\{\lambda_1, \ldots, \lambda_n\}$, $\Delta = diag\{\delta_1, \ldots, \delta_n\}$, $H_i = diag\{h_{i1}, \ldots, h_{in}\}(i = 1, \ldots, 4)$, positive definite matrices $\mathcal{R} = [R_{ij}]_{7\times7} \in \mathbb{R}^{7n\times7n}$, $\mathcal{N} = [N_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{M} = [M_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{G} = [G_{ij}]_{2\times2} \in \mathbb{R}^{2n\times2n}$, $\mathcal{Q}_1 = [Q_{1,ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{Q}_2 = [Q_{2,ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $Q_i(i = 3, \ldots, 8)$, symmetric matrices $P_i(i = 1, \ldots, 3)$, and any matrix $\mathcal{S} = [S_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, satisfying the following LMIs:*

$$(\Gamma^\perp)^T\{\Sigma_1 + \Upsilon\}\Gamma^\perp < 0, \tag{14}$$

$$\begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix} > 0, \tag{15}$$

$$\begin{bmatrix} Q_5 & P_1 \\ \star & Q_6 \end{bmatrix} > 0, \tag{16}$$

$$\begin{bmatrix} Q_7 & P_2 \\ \star & Q_8 \end{bmatrix} > 0, \tag{17}$$

$$\begin{bmatrix} Q_7 & P_3 \\ \star & Q_8 \end{bmatrix} > 0, \tag{18}$$

where Σ_1, Υ, and Γ are defined in (13), and $\Gamma^\perp$ is the right orthogonal complement of Γ.

Proof. For positive diagonal matrices Λ, Δ and positive definite matrices $\mathcal{R}, \mathcal{N}, \mathcal{M}, \mathcal{G}, \mathcal{Q}_1, \mathcal{Q}_2, Q_i(i = 3, \ldots, 8)$ let us consider the following Lyapunov–Krasovskii's functional candidate $V = \sum_{i=1}^{6} V_i$ where

$$V_1 = \begin{bmatrix} x(t) \\ x(t-h_L) \\ x(t-h_U) \\ \int_{t-h_L}^{t} x(s)ds \\ \int_{t-h_U}^{t-h_L} x(s)ds \\ \int_{t-h_L}^{t} f(x(s))ds \\ \int_{t-h_U}^{t-h_L} f(x(s))ds \end{bmatrix}^T \mathcal{R} \begin{bmatrix} x(t) \\ x(t-h_L) \\ x(t-h_U) \\ \int_{t-h_L}^{t} x(s)ds \\ \int_{t-h_U}^{t-h_L} x(s)ds \\ \int_{t-h_L}^{t} f(x(s))ds \\ \int_{t-h_U}^{t-h_L} f(x(s))ds \end{bmatrix},$$

$$V_2 = 2\sum_{i=1}^{n}\left(\int_0^{x_i(t)} (\lambda_i f_i(s) + \delta_i(k_i^+ s - f_i(s)))ds\right),$$

$$V_3 = \int_{t-h_L}^{t} \alpha^T(s)\mathcal{N}\alpha(s)ds + \int_{t-h_U}^{t-h_L} \alpha^T(s)\mathcal{M}\alpha(s)ds + \int_{t-h(t)}^{t} \beta(s)^T\mathcal{G}\beta(s)ds,$$

$$V_4 = h_L\int_{t-h_L}^{t}\int_s^t \alpha^T(u)\mathcal{Q}_1\alpha(u)duds + (h_U - h_L)\int_{t-h_U}^{t-h_L}\int_s^t \alpha^T(u)\mathcal{Q}_2\alpha(u)duds,$$

$$V_5 = (h_L^2/2)\int_{t-h_L}^{t}\int_s^t\int_u^t \dot{x}^T(v)Q_3\dot{x}(v)dvduds, +((h_U^2 - h_L^2)/2)\int_{t-h_U}^{t-h_L}\int_s^t\int_u^t \dot{x}^T(v)Q_4\dot{x}(v)dvduds,$$

$$V_6 = \int_{t-h_L}^{t}\int_s^t \left(x^T(u)Q_5x(u) + \dot{x}^T(u)Q_6\dot{x}(u)\right)duds + \int_{t-h_U}^{t-h_L}\int_s^t \left(x^T(u)Q_7x(u) + \dot{x}^T(u)Q_8\dot{x}(u)\right)duds. \quad (19)$$

The time-derivative of V_1 is calculated as

$$\dot{V}_1 = 2\begin{bmatrix} x(t) \\ x(t-h_L) \\ x(t-h_U) \\ \int_{t-h_L}^{t} x(s)ds \\ \int_{t-h_U}^{t-h_L} x(s)ds \\ \int_{t-h_L}^{t} f(x(s))ds \\ \int_{t-h_U}^{t-h_L} f(x(s))ds \end{bmatrix}^T \mathcal{R} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-h_L) \\ \dot{x}(t-h_U) \\ x(t) - x(t-h_L) \\ x(t-h_L) - x(t-h_U) \\ f(x(t)) - f(x(t-h_L)) \\ f(x(t-h_L)) - f(x(t-h_U)) \end{bmatrix} = \zeta^T(t)\left(\Pi_1\mathcal{R}\Pi_2^T + \Pi_2\mathcal{R}\Pi_1^T\right)\zeta(t). \quad (20)$$

By calculation of $\dot{V}_2$, we have

$$\dot{V}_2 = 2f^T(x(t))\Lambda\dot{x}(t) + 2\left[K_px(t) - f(x(t))\right]^T\Delta\dot{x}(t) = \zeta^T(t)\Phi\zeta(t). \quad (21)$$

With the condition $\dot{h}(t) \leqslant h_D$, an upper bound of V_3 is obtained as

$$\begin{aligned}\dot{V}_3 &\leqslant \alpha^T(t)\mathcal{N}\alpha(t) - \alpha^T(t-h_L)\mathcal{N}\alpha(t-h_L) + \alpha^T(t-h_L)\mathcal{M}\alpha(t-h_L) - \alpha^T(t-h_U)\mathcal{M}\alpha(t-h_U) + \beta^T(t)\mathcal{G}\beta(t) \\ &\quad -(1-h_D)\beta^T(t-h(t))\mathcal{G}\beta(t-h(t)) = \zeta^T(t)\left[\Pi_3\mathcal{N}\Pi_3^T + \Pi_4(-\mathcal{N}+\mathcal{M})\Pi_4^T - \Pi_5\mathcal{M}\Pi_5^T + \Pi_6\mathcal{G}\Pi_6^T - (1-h_D)\Pi_7\mathcal{G}\Pi_7^T\right]\zeta(t). \quad (22)\end{aligned}$$

By use of Lemma 1 and Theorem 1 in [40], an estimation of $\dot{V}_4$ is

$$\begin{aligned}\dot{V}_4 &= h_L^2\alpha^T(t)\mathcal{Q}_1\alpha(t) - h_L\int_{t-h_L}^{t}\alpha^T(s)\mathcal{Q}_1\alpha(s)ds + (h_U-h_L)^2\alpha^T(t)\mathcal{Q}_2\alpha(t) - (h_U-h_L)\int_{t-h(t)}^{t-h_L}\alpha^T(s)\mathcal{Q}_2\alpha(s)ds \\ &\quad - (h_U-h_L)\int_{t-h_U}^{t-h(t)}\alpha^T(s)\mathcal{Q}_2\alpha(s)ds \\ &\leqslant h_L^2\alpha^T(t)\mathcal{Q}_1\alpha(t) - \left(\int_{t-h_L}^{t}\alpha(s)ds\right)^T\mathcal{Q}_1\left(\int_{t-h_L}^{t}\alpha(s)ds\right) + (h_U-h_L)^2\alpha^T(t)\mathcal{Q}_2\alpha(t) \\ &\quad - \left(\frac{h_U-h_L}{h(t)-h_L}\right)\left(\int_{t-h(t)}^{t-h_L}\alpha(s)ds\right)^T\mathcal{Q}_2\left(\int_{t-h(t)}^{t-h_L}\alpha(s)ds\right) - \left(\frac{h_U-h_L}{h_U-h(t)}\right)\left(\int_{t-h_U}^{t-h(t)}\alpha(s)ds\right)^T\mathcal{Q}_2\left(\int_{t-h_U}^{t-h(t)}\alpha(s)ds\right) \\ &\leqslant h_L^2\alpha^T(t)\mathcal{Q}_1\alpha(t) - \left(\int_{t-h_L}^{t}\alpha(s)ds\right)^T\mathcal{Q}_1\left(\int_{t-h_L}^{t}\alpha(s)ds\right) + (h_U-h_L)^2\alpha^T(t)\mathcal{Q}_2\alpha(t) \\ &\quad - \begin{bmatrix}\int_{t-h(t)}^{t-h_L}\alpha(s)ds \\ \int_{t-h_U}^{t-h(t)}\alpha(s)ds\end{bmatrix}^T\begin{bmatrix}\mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2\end{bmatrix}\begin{bmatrix}\int_{t-h(t)}^{t-h_L}\alpha(s)ds \\ \int_{t-h_U}^{t-h(t)}\alpha(s)ds\end{bmatrix} \\ &= \zeta^T(t)\left\{h_L^2\Pi_3\mathcal{Q}_1\Pi_3^T - \Pi_8\mathcal{Q}_1\Pi_8^T + (h_U-h_L)^2\Pi_3\mathcal{Q}_2\Pi_3^T - \Pi_9\begin{bmatrix}\mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2\end{bmatrix}\Pi_9^T\right\}\zeta(t). \quad (23)\end{aligned}$$

By Lemma 2, $\dot{V}_5$ is bounded as

$$\begin{aligned}\dot{V}_5 &= (h_L^2/2)^2\dot{x}^T(t)Q_3\dot{x}(t) - (h_L^2/2)\int_{t-h_L}^{t}\int_s^t \dot{x}^T(u)Q_3\dot{x}(u)duds + ((h_U^2-h_L^2)/2)^2\dot{x}^T(t)Q_4\dot{x}(t) \\ &\quad - \left((h_U^2-h_L^2)/2\right)\int_{t-h_U}^{t-h_L}\int_s^t \dot{x}^T(u)Q_4\dot{x}(u)duds \\ &\leqslant (h_L^2/2)^2\dot{x}^T(t)Q_3\dot{x}(t) - \left(\int_{t-h_L}^{t}\int_s^t \dot{x}(u)duds\right)^T Q_3\left(\int_{t-h_L}^{t}\int_s^t \dot{x}(u)duds\right) \\ &\quad + ((h_U^2-h_L^2)/2)^2\dot{x}^T(t)Q_4\dot{x}(t) - \left(\int_{t-h_U}^{t-h_L}\int_s^t \dot{x}(u)duds\right)^T Q_4\left(\int_{t-h_U}^{t-h_L}\int_s^t \dot{x}(u)duds\right) = \zeta^T(t)\Xi\zeta(t). \end{aligned} \quad (24)$$

Calculation of $\dot{V}_6$ leads to

$$\begin{aligned}\dot{V}_6 &= h_L x^T(t)Q_5x(t) - \int_{t-h_L}^{t} x^T(s)Q_5x(s)ds + h_L\dot{x}^T(t)Q_6\dot{x}(t) - \int_{t-h_L}^{t}\dot{x}^T(s)Q_6\dot{x}(s)ds + (h_U-h_L)x^T(t)Q_7x(t) \\ &\quad - \int_{t-h_U}^{t-h_L} x^T(s)Q_7x(s)ds + (h_U-h_L)\dot{x}^T(t)Q_8\dot{x}(t) - \int_{t-h_U}^{t-h_L}\dot{x}^T(s)Q_8\dot{x}(s)ds. \end{aligned} \quad (25)$$

Inspired by the work of [42], the following three zero equality with any symmetric matrices P_1, P_2, and P_2 are considered:

$$\begin{aligned} 0 &= x^T(t)P_1x(t) - x^T(t-h_L)P_1x(t-h_L) - 2\int_{t-h_L}^{t} x^T(s)P_1\dot{x}(s)ds, \\ 0 &= x^T(t-h_L)P_2x(t-h_L) - x^T(t-h(t))P_2x(t-h(t)) - 2\int_{t-h(t)}^{t-h_L} x^T(s)P_2\dot{x}(s)ds, \\ 0 &= x^T(t-h(t))P_3x(t-h(t)) - x^T(t-h_U)P_3x(t-h_U) - 2\int_{t-h_U}^{t-h(t)} x^T(s)P_3\dot{x}(s)ds. \end{aligned} \quad (26)$$

With the above three zero equalities, an upper bound of $\dot{V}_6$ is

$$\begin{aligned}\dot{V}_6 &\leqslant \zeta^T(t)\Psi\zeta(t) - \int_{t-h_L}^{t}\begin{bmatrix} x(s) \\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix} Q_5 & P_1 \\ \star & Q_6\end{bmatrix}\begin{bmatrix} x(s) \\ \dot{x}(s)\end{bmatrix}ds - \int_{t-h(t)}^{t-h_L}\begin{bmatrix} x(s) \\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix} Q_7 & P_2 \\ \star & Q_8\end{bmatrix}\begin{bmatrix} x(s) \\ \dot{x}(s)\end{bmatrix}ds \\ &\quad - \int_{t-h_U}^{t-h(t)}\begin{bmatrix} x(s) \\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix} Q_7 & P_3 \\ \star & Q_8\end{bmatrix}\begin{bmatrix} x(s) \\ \dot{x}(s)\end{bmatrix}ds. \end{aligned} \quad (27)$$

From (7), for any positive diagonal matrices $H_1 = diag\{h_{11},\ldots,h_{1n}\}$, $H_2 = diag\{h_{21},\ldots,h_{2n}\}$, $H_3 = diag\{h_{31},\ldots,h_{3n}\}$, and $H_4 = diag\{h_{41},\ldots,h_{4n}\}$, the following inequality holds

$$\begin{aligned} 0 &\leqslant -2\sum_{i=1}^{n} h_{1i}\left[f_i(x_i(t)) - k_i^- x_i(t)\right]\left[f_i(x_i(t)) - k_i^+ x_i(t)\right] - 2\sum_{i=1}^{n} h_{2i}\left[f_i(x_i(t-h(t))) - k_i^- x_i(t-h(t))\right] \\ &\quad \times \left[f_i(x_i(t-h(t))) - k_i^+ x_i(t-h(t))\right] - 2\sum_{i=1}^{n} h_{3i}\left[f_i(x_i(t-h_L)) - k_i^- x_i(t-h_L)\right]\left[f_i(x_i(t-h_L)) - k_i^+ x_i(t-h_L)\right] \\ &\quad - 2\sum_{i=1}^{n} h_{4i}\left[f_i(x_i(t-h_U)) - k_i^- x_i(t-h_U)\right]\left[f_i(x_i(t-h_U)) - k_i^+ x_i(t-h_U)\right] = \zeta^T(t)\Upsilon\zeta(t). \end{aligned} \quad (28)$$

From Eqs. (19)–(28) and by application of S-procedure [44], if Eqs. (16)–(18) hold, then an upper bound of $\dot{V}$ is

$$\dot{V} \leqslant \zeta^T(t)\{\Sigma_1 + \Upsilon\}\zeta(t), \quad (29)$$

where Σ_1 are defined in (13).

By Lemma 2, $\zeta^T(t)(\Sigma_1 + \Upsilon)\zeta(t) < 0$ with $0 = \Gamma\zeta(t)$ is equivalent to $(\Gamma^\perp)^T(\Sigma_1 + \Upsilon)\Gamma^\perp < 0$. Therefore, if LMIs (14)–(18) hold, then the neural networks (4) is asymptotically stable. This completes our proof. □

Remark 1. In Theorem 1, the augmented vector $\zeta(t)$ has integrating terms of activation function $f(x(t))$ which are $\int_{t-h_L}^{t} f(x(s))ds$, $\int_{t-h(t)}^{t-h_L} f(x(s))ds$ and $\int_{t-h_U}^{t-h(t)} f(x(s))ds$. By taking these integral terms as augmented vector which has not been considered in other literature, more past history of $f(x(t))$ can be utilized, which may lead less conservative results.

Remark 2. Recently, the reciprocally convex optimization technique to reduce the conservatism of stability criteria for systems with time-varying delays was proposed in [40]. Motivated by this work, the proposed method of [40] was utilized in Eq. (23). However, an augmented vector with $\int_{t-h(t)}^{t-h_L} x(s)ds$, $\int_{t-h_U}^{t-h(t)} x(s)ds$, $\int_{t-h(t)}^{t-h_L} f(x(s))ds$, and $\int_{t-h_U}^{t-h(t)} f(x(s))ds$ was used in Theorem 1, which is different from the method of [40].

Remark 3. In the proposed Lyapunov–Krasovskii's functional of Theorem 1, the terms $(h_U - h_L)\int_{t-h_U}^{t-h_L}\int_s^t \alpha^T(u)\mathcal{Q}_2\alpha(u)duds$, $((h_U^2 - h_L^2)/2)\int_{t-h_U}^{t-h_L}\int_s^t\int_u^t \dot{x}^T(v)Q_4\dot{x}(v)dvduds$, and $\int_{t-h_U}^{t-h_L}\int_s^t(x^T(u)Q_7x(u) + \dot{x}^T(u)Q_8\dot{x}(u))duds$ were proposed at each V_4, V_5, and V_6, respectively. In double integral terms, we have $t - h_U \leqslant s \leqslant t - h_L$ and $s \leqslant u \leqslant t$. Also, in triple integral terms, one can confirm $t - h_U \leqslant s \leqslant t - h_L,\ s \leqslant u \leqslant t$, and $u \leqslant v \leqslant t$. Since the parameter s has the integral interval from $t - h_U$ to $t - h_L$, it may be effective that the maximum values of u and v are changed as $t - h_L$ instead of t. With this regard, in Theorem 2, the terms $(h_U - h_L)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}\alpha^T(u)\mathcal{Q}_2\alpha(u)duds$, $((h_U - h_L)^2/2)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}\int_u^{t-h_L}\dot{x}^T(v)Q_4\dot{x}(v)dvduds$, and $\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}(x^T(u)Q_7x(u) + \dot{x}^T(u)Q_8\dot{x}(u))duds$ will proposed.

Next, based on the results of Theorem 1 and Remark 3, improved stability criteria for system (1) will be introduced as Theorem 2 by taking different interval of integral terms of $V_i(i = 4, 5, 6)$. The notations for some matrices which will be utilized in Theorem 2 are defined as

$$
\begin{aligned}
&\Pi_a = [e_3 \quad e_6 \quad e_{13}],\\
&\widetilde{\Xi} = (h_L^2/2)^2 e_5Q_3e_5^T - (h_Le_1 - e_8)Q_3(h_Le_1 - e_8)^T + \left((h_U - h_L)^2/2\right)^2 e_6Q_4e_6^T - ((h_U - h_L)e_3 - e_9 - e_{10})Q_4((h_U - h_L)e_3 - e_9 - e_{10})^T,\\
&\widetilde{\Psi} = h_Le_1Q_5e_1^T + h_Le_5Q_6e_5^T + (h_U - h_L)e_3Q_7e_3^T + (h_U - h_L)e_6Q_8e_6^T + e_1P_1e_1^T + e_3(-P_1 + P_2)e_3^T + e_2(-P_2 + P_3)e_2^T - e_4P_3e_4^T,\\
&\Sigma_2 = \widetilde{\Xi} + \Phi + \widetilde{\Psi} + \Pi_1\mathcal{R}\Pi_2^T + \Pi_2\mathcal{R}\Pi_1^T + \Pi_3\mathcal{N}\Pi_3^T + \Pi_4(-\mathcal{N} + \mathcal{M})\Pi_4^T - \Pi_5\mathcal{M}\Pi_5^T + \Pi_6\mathcal{G}\Pi_6^T - (1 - h_D)\Pi_7\mathcal{G}\Pi_7^T + h_L^2\Pi_3\mathcal{Q}_1\Pi_3^T\\
&\qquad - \Pi_8\mathcal{Q}_1\Pi_8^T + (h_U - h_L)^2\Pi_a\mathcal{Q}_2\Pi_a^T - \Pi_9\begin{bmatrix}\mathcal{Q}_2 & \mathcal{S}\\ \star & \mathcal{Q}_2\end{bmatrix}\Pi_9^T.
\end{aligned}
\tag{30}
$$

and other notations expressed in Σ_2 are the same ones as in Eq. (13).

Now, we have the following theorem.

Theorem 2. *For given positive scalars h_L and h_U, any scalar h_D, diagonal matrices $K_m = diag\{k_1^-, \ldots, k_n^-\}$ and $K_p = diag\{k_1^+, \ldots, k_n^+\}$, the system (1) is asymptotically stable for $0 \leqslant h_L \leqslant h(t) \leqslant h_U$ and $\dot{h}(t) \leqslant h_D$ if there exist positive diagonal matrices $\Lambda = diag\{\lambda_1, \ldots, \lambda_n\}$, $\Delta = diag\{\delta_1, \ldots, \delta_n\}$, $H_i = diag\{h_{i1}, \ldots, h_{in}\}(i = 1, \ldots, 4)$, positive definite matrices $\mathcal{R} = [R_{ij}]_{7\times7} \in \mathbb{R}^{7n\times7n}$, $\mathcal{N} = [N_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{M} = [M_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{G} = [G_{ij}]_{2\times2} \in \mathbb{R}^{2n\times2n}$, $\mathcal{Q}_1 = [Q_{1,ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{Q}_2 = [Q_{2,ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $Q_i(i = 3, \ldots, 8)$, symmetric matrices $P_i(i = 1, \ldots, 3)$, and any matrix $\mathcal{S} = [S_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, satisfying the following LMIs:*

$$(\Gamma^\perp)^T\{\Sigma_2 + \Upsilon\}\Gamma^\perp < 0, \tag{31}$$

$$\begin{bmatrix}\mathcal{Q}_2 & \mathcal{S}\\ \star & \mathcal{Q}_2\end{bmatrix} > 0, \tag{32}$$

$$\begin{bmatrix}Q_5 & P_1\\ \star & Q_6\end{bmatrix} > 0, \tag{33}$$

$$\begin{bmatrix}Q_7 & P_2\\ \star & Q_8\end{bmatrix} > 0, \tag{34}$$

$$\begin{bmatrix}Q_7 & P_3\\ \star & Q_8\end{bmatrix} > 0, \tag{35}$$

where Σ_2 was defined in Eq. (30).

Proof. For positive diagonal matrices Λ, Δ and positive definite matrices $\mathcal{R}, \mathcal{N}, \mathcal{M}, \mathcal{G}, \mathcal{Q}_1, \mathcal{Q}_2, Q_i(i = 3, \ldots, 8)$ let us consider the following Lyapunov–Krasovskii's functional candidate $V = \sum_{i=1}^6 V_i$ where

$$
\begin{aligned}
V_4 &= h_L\int_{t-h_L}^t\int_s^t \alpha^T(u)\mathcal{Q}_1\alpha(u)duds + (h_U - h_L)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}\alpha^T(u)\mathcal{Q}_2\alpha(u)duds,\\
V_5 &= (h_L^2/2)\int_{t-h_L}^t\int_s^t\int_u^t \dot{x}^T(v)Q_3\dot{x}(v)dvduds, + ((h_U - h_L)^2/2)\int_{t-h_U}^{t-h_L}\int_s^{t-h_L}\int_u^{t-h_L}\dot{x}^T(v)Q_4\dot{x}(v)dvduds,\\
V_6 &= \int_{t-h_L}^t\int_s^t\left(x^T(u)Q_5x(u) + \dot{x}^T(u)Q_6\dot{x}(u)\right)duds + \int_{t-h_U}^{t-h_L}\int_s^{t-h_L}\left(x^T(u)Q_7x(u) + \dot{x}^T(u)Q_8\dot{x}(u)\right)duds,
\end{aligned}
\tag{36}
$$

and $V_i(i = 1, 2, 3)$ are the same ones as in Eq. (19).

By using the similar method presented at Eq. (23), an upper bound of time-derivative of $\dot{V}_4$ can be

$$
\begin{aligned}
\dot{V}_4 &\leqslant h_L^2\alpha^T(t)\mathcal{Q}_1\alpha(t) - h_L\int_{t-h_L}^t\alpha^T(s)\mathcal{Q}_1\alpha(s)ds + (h_U - h_L)^2\alpha^T(t - h_L)\mathcal{Q}_2\alpha(t - h_L)\\
&\quad - (h_U - h_L)\int_{t-h(t)}^{t-h_L}\alpha^T(s)\mathcal{Q}_2\alpha(s)ds - (h_U - h_L)\int_{t-h_U}^{t-h(t)}\alpha^T(s)\mathcal{Q}_2\alpha(s)ds\\
&\leqslant \zeta^T(t)\left\{h_L^2\Pi_3\mathcal{Q}_1\Pi_3^T - \Pi_8\mathcal{Q}_1\Pi_8^T + (h_U - h_L)^2\Pi_a\mathcal{Q}_2\Pi_a^T - \Pi_9\begin{bmatrix}\mathcal{Q}_2 & \mathcal{S}\\ \star & \mathcal{Q}_2\end{bmatrix}\Pi_9^T\right\}\zeta(t).
\end{aligned}
\tag{37}
$$

By calculating $\dot{V}_5$, we have

$$\begin{aligned}\dot{V}_5 &= (h_L^2/2)^2\dot{x}^T(t)Q_3\dot{x}(t) - (h_L^2/2)\int_{t-h_L}^{t}\int_{s}^{t}\dot{x}^T(u)Q_3\dot{x}(u)duds + ((h_U-h_L)^2/2)^2\dot{x}^T(t-h_L)Q_4\dot{x}(t-h_L)\\ &\quad - \left((h_U-h_L)^2/2\right)\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}\dot{x}^T(u)Q_4\dot{x}(u)duds\\ &\leqslant (h_L^2/2)^2\dot{x}^T(t)Q_3\dot{x}(t) - \left(\int_{t-h_L}^{t}\int_{s}^{t}\dot{x}(u)duds\right)^T Q_3\left(\int_{t-h_L}^{t}\int_{s}^{t}\dot{x}(u)duds\right) + ((h_U-h_L)^2/2)^2\dot{x}^T(t-h_L)Q_4\dot{x}(t-h_L)\\ &\quad - \left(\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}\dot{x}(u)duds\right)^T Q_4\left(\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L}\dot{x}(u)duds\right)\\ &= \zeta^T(t)\widetilde{\Xi}\zeta(t). \end{aligned} \tag{38}$$

From the result of $\dot{V}_6$, one can obtain

$$\begin{aligned}\dot{V}_6 &= h_L x^T(t)Q_5x(t) - \int_{t-h_L}^{t}x^T(s)Q_5x(s)ds + h_L\dot{x}^T(t)Q_6\dot{x}(t) - \int_{t-h_L}^{t}\dot{x}^T(s)Q_6\dot{x}(s)ds + (h_U-h_L)x^T(t-h_L)Q_7x(t-h_L)\\ &\quad - \int_{t-h_U}^{t-h_L}x^T(s)Q_7x(s)ds + (h_U-h_L)\dot{x}^T(t-h_L)Q_8\dot{x}(t-h_L) - \int_{t-h_U}^{t-h_L}\dot{x}^T(s)Q_8\dot{x}(s)ds. \end{aligned} \tag{39}$$

With the obtained results (37)–(39), the other procedure is straightforward from the proof of Theorem 1, so we omit it. □

Remark 4. Since a delay-partitioning idea was firstly proposed in [38], it is well recognized that delay-partitioning approach can increase the feasible region of stability criteria owing to the fact that this method can obtain more tighter upper bounds obtained by calculating the time-derivative of Lyapunov–Krasovskii functional, which leads to less conservative results. However, when the number of delay-partitioning number increases, the matrix formulation becomes more complex and the computational burden and time-consuming grow bigger. Noticing this fact mentioned above, in Theorem 3, the bounding of activation function $k_i^- \leqslant \frac{f_i(u)}{u} \leqslant k_i^+$ will be divided into two subintervals such as $k_i^- \leqslant \frac{f_i(u)}{u} \leqslant (k_i^- + k_i^+)/2$ and $(k_i^- + k_i^+)/2 \leqslant \frac{f_i(u)}{u} \leqslant k^+$ instead of no using delay-partitioning approach. To the authors' best knowledge, this approach has not been proposed. Through two numerical examples, it will be shown Theorem 3 significantly improves the feasible region of stability criterion comparing with those of Theorem 2.

Finally, based on the results of Theorem 2 and Remark 4 mentioned above, a novel approach for delay-range-dependent stability criterion for system (4) will be introduced. For the sake of simplicity in matrix representation, the notations for some matrices of Theorem 3 are defined as

$$\begin{aligned}\Upsilon_a &= e_1((3K_m+K_p)/2)H_1e_{11}^T + e_{11}H_1((3K_m+K_p)/2)e_1^T - 2e_{11}H_1e_{11}^T - 2e_1K_mH_1((Km+Kp)/2)e_1^T\\ &\quad + e_2((3K_m+K_p)/2)H_2e_{12}^T + e_{12}H_2((3K_m+K_p)/2)e_2^T - 2e_{12}H_2e_{12}^T - 2e_2K_mH_2((Km+Kp)/2)e_2^T\\ &\quad + e_3((3K_m+K_p)/2)H_3e_{13}^T + e_{13}H_3((3K_m+K_p)/2)e_3^T - 2e_{13}H_3e_{13}^T - 2e_3K_mH_3((Km+Kp)/2)e_3^T\\ &\quad + e_4((3K_m+K_p)/2)H_4e_{14}^T + e_{14}H_4((3K_m+K_p)/2)e_4^T - 2e_{14}H_4e_{14}^T - 2e_4K_mH_4((Km+Kp)/2)e_4^T,\\ \Upsilon_b &= e_1((K_m+3K_p)/2)H_5e_{11}^T + e_{11}H_5((K_m+3K_p)/2)e_1^T - 2e_{11}H_5e_{11}^T - 2e_1((Km+Kp)/2)H_5K_pe_1^T\\ &\quad + e_2((K_m+3K_p)/2)H_6e_{12}^T + e_{12}H_6((K_m+3K_p)/2)e_2^T - 2e_{12}H_6e_{12}^T - 2e_2((Km+Kp)/2)H_6K_pe_2^T\\ &\quad + e_3((K_m+3K_p)/2)H_7e_{13}^T + e_{13}H_7((K_m+3K_p)/2)e_3^T - 2e_{13}H_7e_{13}^T - 2e_3((Km+Kp)/2)H_7K_pe_3^T\\ &\quad + e_4((K_m+3K_p)/2)H_8e_{14}^T + e_{14}H_8((K_m+3K_p)/2)e_4^T - 2e_{14}H_8e_{14}^T - 2e_4((Km+Kp)/2)H_8K_pe_4^T. \end{aligned} \tag{40}$$

Now, we have the following theorem.

Theorem 3. *For given positive scalars h_L and h_U, any scalar h_D, diagonal matrices $K_m = diag\{k_1^-,\ldots,k_n^-\}$ and $K_p = diag\{k_1^+,\ldots,k_n^+\}$, the system (1) is asymptotically stable for $0 \leqslant h_L \leqslant h(t) \leqslant h_U$ and $\dot{h}(t) \leqslant h_D$ if there exist positive diagonal matrices $\Lambda = diag\{\lambda_1,\ldots,\lambda_n\}$, $\Delta = diag\{\delta_1,\ldots,\delta_n\}$, $H_i = diag\{h_{i1},\ldots,h_{in}\}(i=1,\ldots,8)$, positive definite matrices $\mathcal{R} = [R_{ij}]_{7\times7} \in \mathbb{R}^{7n\times7n}$, $\mathcal{N} = [N_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{M} = [M_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{G} = [G_{ij}]_{2\times2} \in \mathbb{R}^{2n\times2n}$, $\mathcal{Q}_1 = [Q_{1,ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $\mathcal{Q}_2 = [Q_{2,ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, $Q_i(i=3,\ldots,8)$, symmetric matrices $P_i(i=1,\ldots,3)$, and any matrix $\mathcal{S} = [S_{ij}]_{3\times3} \in \mathbb{R}^{3n\times3n}$, satisfying the following LMIs:*

$$(\Gamma^{\perp})^T\{\Sigma_2 + \Upsilon_a\}\Gamma^{\perp} < 0, \tag{41}$$

$$(\Gamma^{\perp})^T\{\Sigma_2 + \Upsilon_b\}\Gamma^{\perp} < 0, \tag{42}$$

$$\begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix} > 0, \tag{43}$$

$$\begin{bmatrix} Q_5 & P_1 \\ \star & Q_6 \end{bmatrix} > 0, \tag{44}$$

$$\begin{bmatrix} Q_7 & P_2 \\ \star & Q_8 \end{bmatrix} > 0, \tag{45}$$

$$\begin{bmatrix} Q_7 & P_3 \\ \star & Q_8 \end{bmatrix} > 0, \tag{46}$$

where Σ_2, is defined in (30), Υ_a, and Υ_b are in Eq. (40).

Proof. For positive diagonal matrices Λ, Δ and positive definite atrices $\mathcal{R}, \mathcal{N}, \mathcal{M}, \mathcal{G}, \mathcal{Q}_1, \mathcal{Q}_2, Q_i (i = 3, \ldots, 8)$ let us consider the same Lyapunov–Krasovskii functional (36) proposed in Theorem 2.

Case I: $k_i^- \leqslant \frac{f_i(u)}{u} \leqslant (k_i^- + k_i^+)/2$. From (7), for any positive diagonal matrices $H_1 = diag\{h_{11}, \ldots, h_{1n}\}$, $H_2 = diag\{h_{21}, \ldots, h_{2n}\}$, $H_3 = diag\{h_{31}, \ldots, h_{3n}\}$, and $H_4 = diag\{h_{41}, \ldots, h_{4n}\}$, the following inequality holds

$$\begin{aligned} 0 \leqslant & -2\sum_{i=1}^{n} h_{1i}\left[f_i(x_i(t)) - k_i^- x_i(t)\right]\left[f_i(x_i(t)) - ((k_i^- + k_i^+)/2)x_i(t)\right] \\ & -2\sum_{i=1}^{n} h_{2i}\left[f_i(x_i(t-h(t))) - k_i^- x_i(t-h(t))\right]\left[f_i(x_i(t-h(t))) - ((k_i^- + k_i^+)/2)x_i(t-h(t))\right] \\ & -2\sum_{i=1}^{n} h_{3i}\left[f_i(x_i(t-h_L)) - k_i^- x_i(t-h_L)\right]\left[f_i(x_i(t-h_L)) - ((k_i^- + k_i^+)/2)x_i(t-h_L)\right] \\ & -2\sum_{i=1}^{n} h_{4i}\left[f_i(x_i(t-h_U)) - k_i^- x_i(t-h_U)\right]\left[f_i(x_i(t-h_U)) - ((k_i^- + k_i^+)/2)x_i(t-h_U)\right] = \zeta^T(t)\Upsilon_a\zeta(t). \end{aligned} \tag{47}$$

Then, from the proof of Theorem 1 and 2, when $k_i^- \leqslant \frac{f_i(u)}{u} \leqslant (k_i^- + k_i^+)/2$, an upper bound of $\dot{V}$ can be

$$\dot{V} \leqslant \zeta^T(t)\{\Sigma_2 + \Upsilon_a\}\zeta(t) \tag{48}$$

with $0 = \Gamma\zeta(t)$. Therefore, from Lemma 2 and S-procedure [44], if (41), (42), (42), (43, (46) hold, then system (4) is asymptotically stable for $0 \leqslant h_L \leqslant h(t) \leqslant h_U,\ \dot{h}(t) \leqslant h_D$, and $k_i^- \leqslant \frac{f_i(u)}{u} \leqslant (k_i^- + k_i^+)/2$.

Case II: $(k_i^- + k_i^+)/2 \leqslant \frac{f_i(u)}{u} \leqslant k_i^+$.

Note that the condition $k_i^+/2 \leqslant \frac{f_i(u)}{u} \leqslant k_i^+$ is equivalent to

$$\left[f_i(u) - ((k_i^- + k_i^+)/2)u\right]\left[f_i(u) - k_i^+ u\right] < 0, \quad i = 1, \ldots, n. \tag{49}$$

From (49), for any positive diagonal matrices $H_5 = diag\{h_{51}, \ldots, h_{5n}\}$, $H_6 = diag\{h_{61}, \ldots, h_{6n}\}$, $H_7 = diag\{h_{71}, \ldots, h_{7n}\}$, and $H_8 = diag\{h_{81}, \ldots, h_{8n}\}$, the following inequality holds

$$\begin{aligned} 0 \leqslant & -2\sum_{i=1}^{n} h_{5i}\left[f_i(x_i(t)) - ((k_i^- + k_i^+)/2)x_i(t)\right]\left[f_i(x_i(t)) - k_i^+ x_i(t)\right] \\ & -2\sum_{i=1}^{n} h_{6i}\left[f_i(x_i(t-h(t))) - ((k_i^- + k_i^+)/2)x_i(t-h(t))\right]\left[f_i(x_i(t-h(t))) - k_i^+ x_i(t-h(t))\right] \\ & -2\sum_{i=1}^{n} h_{7i}\left[f_i(x_i(t-h_L)) - ((k_i^- + k_i^+)/2)x_i(t-h_L)\right]\left[f_i(x_i(t-h_L)) - k_i^+ x_i(t-h_L)\right] \\ & -2\sum_{i=1}^{n} h_{8i}\left[f_i(x_i(t-h_U)) - ((k_i^- + k_i^+)/2)x_i(t-h_U)\right]\left[f_i(x_i(t-h_U)) - k_i^+ x_i(t-h_U)\right] = \zeta^T(t)\Upsilon_b\zeta(t). \end{aligned} \tag{50}$$

Therefore, from Lemma 2 and S-procedure [44], if (42), (42), (42), (43, (46) hold, then system (4) is asymptotically stable for $0 \leqslant h_L \leqslant h(t) \leqslant h_U,\ \dot{h}(t) \leqslant h_D$, and $(k_i^- + k_i^+)/2 \leqslant \frac{f_i(u)}{u} \leqslant k_i^+$. Thus, the feasibility of (42), (42), (42), (43, (46) means that system (4) is asymptotically stable for $0 \leqslant h_L \leqslant h(t) \leqslant h_U,\ \dot{h}(t) \leqslant h_D$, and $k_i^- \leqslant \frac{f_i(u)}{u} \leqslant k_i^+$. This completes the proof of Theorem 3. □

Remark 5. When the information of an upper bound of $\dot{h}(t)$ is unknown or larger than one, Theorem 1, 2 also can provide delay-dependent stability criteria for (1) by letting $\mathcal{G} = 0$.

4. Numerical examples

Example 1. Consider the neural networks (4) where

Table 1
Delay bounds h_U with $h_L = 3$ and different h_D (Example 1).

h_D	0.1	0.5	0.9	Unknown (or $h_D \geqslant 1$)
[30]$(m = 2)^*$	3.65	3.32	3.26	3.24
[30]$(m = 4)^*$	3.71	3.36	3.29	3.28
[36]$(m = 2)^*$	3.78	3.45	3.39	3.38
Theorem 1	4.0071	3.3960	3.3033	3.2827
Theorem 2	4.0130	3.4470	3.3403	3.3196
Theorem 3	4.1967	3.6246	3.5961	3.5952

$*$ m is delay-partitioning number.

$$A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix},$$
$$W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$
$$W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$
$$K_p = diag\{0.1137, 0.1279, 0.7994, 0.2368\}. \tag{51}$$

For this system, when $h_L = 3$, by dividing the lower bound of the time-varying delay, improved delay-dependent stability criterion was proposed in [30]. Very recently, with the consideration of the lower bound of delay derivative and delay-partitioning technique, less conservative results were presented in [36]. Table 1 gives the comparison results on the maximum delay bound allowed via the methods in recent works and our new study. From Table 1, it can be seen that Theorem 2

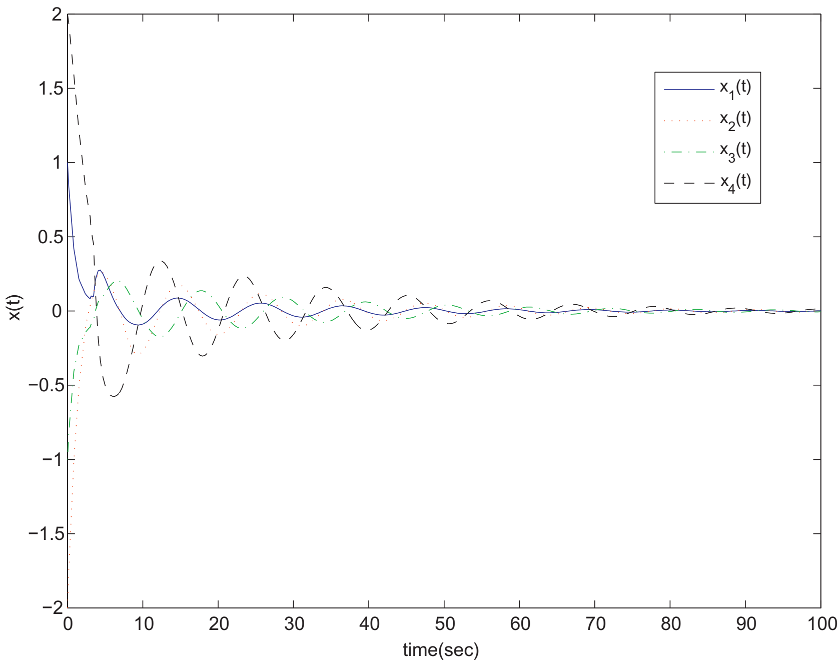

Fig. 1. State trajectories of system (51) when $h(t) = 3 + 0.59|sin(10t)|$.

Table 2
Delay bounds h_U with different h_D (Example 2).

		$h_D = 0.8$	$h_D = 0.9$	Unknown h_D(or $h_D \geqslant 1$)
[23]	$h_L = 1$	2.5967	2.0443	1.9621
[32]		3.8359	2.9234	2.7532
Theorem 1		4.8278	3.6889	3.2975
Theorem 2		4.8278	3.6889	3.2975
Theorem 3		4.8668	3.8047	3.6001
[23]	$h_L = 100$	101.5946	101.0443	100.9621
[32]		102.8335	101.9234	101.7532
Theorem 1		103.7774	102.6887	102.2975
Theorem 2		103.7776	102.6887	102.2975
Theorem 3		103.8101	102.8047	102.6001

provides larger delay bounds than those of Theorem 1, which supports the effectiveness of the proposed idea introduced in Theorem 2. Note that Theorem 1 and 2 give larger delay bounds than those of [30] but fails the improvement of the feasible region when h_D is 0.5, 0.9, and unknown comparing with the results of [36]. However, Theorem 3 significantly reduces the conservatism of Theorem 1 and 2 and provides larger delay bounds than the existing ones of [36] in spite of no utilizing delay-partitioning techniques. To confirm the obtained results of Theorem 3 when $h(t)$ is unknown, the state trajectories of system (51) when $h(t) = 3 + 0.59|sin(10t)|$ are shown in Fig. 1.

Example 2. Consider the neural networks (4) with the parameters

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$
$$W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \quad K_p = diag\{0.4, 0.8\},$$
$$K_m = diag\{0, 0\}. \tag{52}$$

For this system, when $h_L = 1$ and $h_L = 100$, the maximum delay bounds obtained by the methods of [23,32] for various conditions of h_D are listed in Table 2. In [32], by dividing delay interval into two and employing different free-weighting matrices at each interval, improved maximum delay bounds were obtained. With the same conditions presented in Table 2, the obtained results by applying Theorem 1, 2 to the above system (52) are also compared with the existing ones of [23,32]. From Table 2, one can see Theorem 1 which does not employ delay partitioning technique provides larger delay bounds than the results of [32]. Even though the results of Theorem 2 do not significantly enhance the feasible region of Theorem 1, Theorem 3 gives larger delay bounds than those of Theorem 2. This supports the effectiveness of the proposed idea in Theorem 3 in reducing the conservatism of stability criteria.

5. Conclusions

In this paper, three delay-dependent stability criteria for neural networks with interval time-varying delays have been proposed by the use of Lyapunov method and LMI framework. In Theorem 1, by constructing the augmented Lyapunov–Krasovskii functional and utilizing reciprocal convex optimization approach introduced in [40], less conservative results of stability criterion has been proposed without the use of delay-partitioning techniques. Based on the results of Theorem 1, it was shown that improved feasible region of stability criterion can be obtained by modifying some intervals of integral terms in the proposed Lyapunov–Krasovskii functional. By dividing the bounding of activation functions into two, the further improved stability criterion was proposed in Theorem 3. Through two well-known examples, the improvement of the proposed stability criteria has been successfully verified.

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