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Design of state estimator for genetic regulatory networks with time-varying delays and randomly occurring uncertainties

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In this paper, the design problem of state estimator for genetic regulatory networks with time delays and randomly occurring uncertainties has been addressed by a delay decomposition approach. The norm-bounded uncertainties enter into the genetic regulatory networks (GRNs) in random ways, and such randomly occurring uncertainties (ROUs) obey certain mutually uncorrelated Bernoulli distributed white noise sequences. Under these circumstances, the state estimator is designed to estimate the true concentration of the mRNA and the protein of the uncertain GRNs. Delay-dependent stability criteria are obtained in terms of linear matrix inequalities by constructing a Lyapunov–Krasovskii functional and using some inequality techniques (LMIs). Then, the desired state estimator, which can ensure the estimation error dynamics to be globally asymptotically robustly stochastically stable, is designed from the solutions of LMIs. Finally, a numerical example is provided to demonstrate the feasibility of the proposed estimation schemes.

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1. Introduction

In the living cell, some of the gene products are synthesized at all time and many other are expressed at some circumstances. These gene products are synthesized through the gene expression which is controlled by a collection of DNA segments called as gene regulatory network (GRN). GRN governs the level and rate of gene expression (gene product) through the feedback and feed forward control regulation system. That is, the increased amount of gene product will inhibit the gene expression and decreased amount of gene product will induce the gene expression through a network of biological molecules. Here, we have demonstrated the outline of GRN through gene on and off system illustrated in Fig. 1 as in Alberts et al. (2009). In the past few decades, the genetic regulatory network attracts the attention of researchers at interdisciplinary field to expand the application of biology (Chauviya et al., 2006; Chen and Weng, 2011; Hoteit et al., 2012). In general, genetic network models can be classified into two types, that is, the Boolean model and the differential equation model. In the Boolean model, the state of a gene is determined by a Boolean function and each gene is shown in one of two states: ON or OFF. In the differential equation model, variables describe the concentration of gene products, such as mRNAs and proteins, as continuous values of the gene regulation systems.

Recent works (Cit. 2007; Gaffney and Monk, 2006; Ribeiro, 2010; Samad et al., 2005) have demonstrated that time delays can play an important role in the biological systems. Due to slow biochemical reactions such as gene transcription and translation, this should be considered. For example, in Orosz et al. (2010), the authors have discussed the controlling biological networks by time-delayed signals. In Mac Donald (1989), the detail explanation about the biological delay systems is provided. In this regard, a great number of results on stability of biological systems have been investigated (see, for example, Balasubramaniam et al., 2010a; Mathiyalagan et al., 2012a,b; Mathiyalagan and Sakhivel, 2012; Sakhivel et al., 2010, 2012; Jin and Lindsey, 2008; Wang et al., 2010a; Wei et al., 2000). When modeling...
dynamical networks, there are some unavoidable things, such as modeling error, external perturbation and parameter fluctuation. These factors would possibly degrade the model quality significantly. In other word, the genetic networks and neural networks model certainly involve uncertainties such as perturbations and component variations, which will change the stability of genetic networks and neural networks and such variations are unknown but with known bounds (Mousavi and Majd, 2011). Therefore, when it considers the genetic networks and neural networks with uncertainties, it is of crucial importance to develop the robust analysis techniques in the networks. The problem of robust stability analysis for GRNs with time delay has been investigated (see for example, Koo et al., 2012; Wang et al., 2010a,b; Wang and Zhong, 2012; Wu, 2011; Wu et al., 2010). Recently, a new type of uncertainty named as randomly occurring uncertainties (ROUs) has been proposed in Lee et al. (2012) and Wu et al. (2012). System uncertainties may be subject to random changes in real circumstances due to some factors such as repairs of components and sudden environmental disturbances and thus it may occur in a probabilistic way with certain types and intensity.

On the other hand, as another real control problem, the state estimation problem of delayed neural networks was studied in Wang et al. (2005). In Ahn (2012), Balasubramaniam et al. (2010b), Wang et al. (2009), and Zheng et al. (2010), the problem has been extended to general Markovian systems, passivity theory, neural networks as well as complex networks. At the same time, an interesting problem of the biological significance is therefore to estimate the true states of GRNs (concentrations of mRNA and proteins). In a relatively large scale or high-order complex system, it is often the case that only partial information about the states of the nodes is available in the network outputs. Hence, it is necessary to estimate the states of the nodes through available measurements. Therefore, one of the hottest research topics is the state estimation problem for GRNs with time-varying delays. In Li et al. (2010), Liang et al. (2009), Liang and Lam (2010) and Lv et al. (2011), the design problem of state estimator for GRNs have been dealt with stochastic disturbances, Markovian switchings and parameter uncertainties. At the same time, the works in Zhu and Yang (2010) and Lakshmanan and Balasubramaniam (2011), Lakshmanan et al. (2012) have provided a less conservative stability condition for delayed systems and neural networks with Markovian jumping parameters by using delay decomposition approaches. In this paper, the integral intervals $[t−\tau, t]$ and $[t−\tau, t]$ are decomposed into

![A) Gene expression on](image)

![B) Gene expression off](image)
2. Problem description and preliminaries

Based on the GRNs model in Yuh et al. (1998), Chen and Aihara (2002), and Li et al. (2006), we consider the following nonlinear GRNs with time-varying delays described by

\[
\begin{align*}
\hat{\mathbf{x}}(t) &= -A\mathbf{x}(t) + B\mathbf{g}(\mathbf{y}(t) - \tau(t))) + \mathbf{f}(t), \\
\hat{\mathbf{y}}(t) &= -C\hat{\mathbf{y}}(t) + D\mathbf{x}(t) - \sigma(t)),
\end{align*}
\]

where \( \mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \), \( \mathbf{y}(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in \mathbb{R}^n \), \( \hat{\mathbf{x}}(t), \hat{\mathbf{y}}(t) \in \mathbb{R} \) are the concentrations of mRNA and protein, respectively; \( A = \text{diag}(a_1, a_2, \ldots, a_n) \) and \( C = \text{diag}(c_1, c_2, \ldots, c_n) \) are constant matrices implying the rates of degradation; \( D = \text{diag}(d_1, d_2, \ldots, d_n) \) represents the transposition rate; \( B = (b_{ij})_{n \times n} \) is the coupling matrix of the genetic networks; the nonlinear function \( \mathbf{g}(\mathbf{y}(t)) = [g_1(\mathbf{y}(t)), g_2(\mathbf{y}(t)), \ldots, g_n(\mathbf{y}(t))]^T \) represents the feedback regulation function of protein on transcription, which is the monotonic function in Hill form, i.e., \( g_i(\mathbf{y}(t)) = \frac{h_i}{1 + \mathbf{y}(t)^T} \) where \( h_i \) is the Hill coefficient; \( f(t) = [f_1(t), f_2(t), \ldots, f_n(t)]^T \) stands for the basal rates of degradation; and the inter-and-intra-node time-varying delays \( \tau(t) \) and \( \sigma(t) \) are assumed to satisfy

\[
0 \leq \tau(t) \leq \tau, \quad 0 \leq \sigma(t) \leq \sigma, \quad \hat{\mathbf{x}}(t) \leq \mu_1, \quad \dot{\sigma}(t) \leq \mu_2.
\]

Since \( g(\cdot) \) is a monotonically increasing function with saturation, from the definition of \( g(\cdot) \), it satisfies the following condition

\[
g(\mathbf{y})g(\mathbf{y}) - I\mathbf{y} \preceq 0,
\]

where \( I = \text{diag}(1, \ldots, n) \).

In relatively large-scale or high-order complex systems, it is normally the case that only partial information about the network components are available. Therefore, in order to obtain the true state of the GRNs (concentrations of the mRNA and protein), one would need to estimate them from available measurements. Similar to references Li et al. (2010), Liang et al. (2009), Liang and Lam (2010) and Lv et al. (2011), here the network measurements are given as follows:

\[
\begin{align*}
\mathbf{z}_1(t) &= M\hat{\mathbf{x}}(t), \\
\mathbf{z}_2(t) &= N\hat{\mathbf{y}}(t)
\end{align*}
\]

where \( \mathbf{z}_1(t), \mathbf{z}_2(t) \in \mathbb{R}^k \) are the actual measurement outputs and \( M, N \) are known constant matrices with appropriate dimensions.

The main objective of this paper is to estimate the concentrations of mRNA and protein in (1) from the available network outputs (3). Here the considered state estimator is of the following form:

\[
\begin{align*}
\hat{\mathbf{x}}(t) &= -A\hat{\mathbf{x}}(t) + B\mathbf{g}(\mathbf{y}(t) - \tau(t))) + \mathbf{f}(t) + K_1[\mathbf{z}_1(t) - M\hat{\mathbf{x}}(t)], \\
\hat{\mathbf{y}}(t) &= -C\hat{\mathbf{y}}(t) + D\mathbf{x}(t) - \sigma(t)) + K_2[\mathbf{z}_2(t) - N\hat{\mathbf{y}}(t)],
\end{align*}
\]

where \( \hat{\mathbf{x}}(t), \hat{\mathbf{y}}(t) \in \mathbb{R}^n \) are the estimation of \( \mathbf{x}(t) \), \( \hat{\mathbf{y}}(t) \) and \( \mathbf{K}_1, \mathbf{K}_2 \in \mathbb{R}^{n \times k} \) are the estimator gain matrices to be determined later.

Our aim is to choose suitable \( \mathbf{K}_1 \) and \( \mathbf{K}_2 \), so that \( \hat{\mathbf{x}}(t) \) and \( \hat{\mathbf{y}}(t) \), respectively, approach to \( \mathbf{x}(t) \) and \( \mathbf{y}(t) \) asymptotically. For this purpose, let the error state vectors be \( \mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) \) and \( \mathbf{y}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t) \); it follows from (1), (3) and (4) that

\[
\begin{align*}
\dot{\mathbf{e}}(t) &= -[A + K_1M]\mathbf{e}(t) + B\mathbf{g}(\mathbf{y}(t) - \tau(t))), \\
\dot{\mathbf{y}}(t) &= -[C + K_2N]\mathbf{e}(t) + D\mathbf{e}(t) - \sigma(t)).
\end{align*}
\]

The initial conditions for the uncertain delayed GRNs (1) and the state estimator system (4) are assumed to be \( \mathbf{e}(t), \mathbf{y}(t) = (\mathbf{e}_0(t), \mathbf{y}_0(t)), \mathbf{e}(t), \mathbf{y}(t) = (\mathbf{e}_0(t), \mathbf{y}_0(t)) \) on \( -\tau \leq t \leq 0 \) and \( \tau^* = \max\{\tau, \sigma\} \).

For simplicity, let \( \mathbf{x}(\tau, \varphi_1, \varphi_1), \mathbf{y}(\tau, \varphi_1, \varphi_1) \) and \( \mathbf{x}(\tau, \varphi_2, \varphi_2), \mathbf{y}(\tau, \varphi_2, \varphi_2) \) be the trajectories of the uncertain delayed GRNs (1) and the state estimator system (4).

For the sake of convenience, the following lemmas are introduced as follows.
Lemma 2.1 ([Zhao et al., 2007]). For any constant matrices $M \in \mathbb{R}^{n \times n}$, scalar $\gamma > 0$, vector function $\dot{\omega} : [0, \gamma] \rightarrow \mathbb{R}^n$ such that the integrations concerned are well defined, then

$$-\gamma \int_{-\gamma}^{0} \dot{\omega}^T(t+s)M\dot{\omega}(t+s)ds \leq -\left( \int_{-\gamma}^{0} \dot{\omega}(t+s)ds \right)^T M \left( \int_{-\gamma}^{0} \dot{\omega}(t+s)ds \right).$$

Rearranging the term $\int_{-\gamma}^{0} \dot{\omega}(t+s)ds$ with $\omega(t) - \omega(t-\gamma)$, we can yield the following inequality

$$-\gamma \int_{-\gamma}^{0} \dot{\omega}^T(t+s)M\dot{\omega}(t+s)ds \leq \begin{bmatrix} \omega(t) \\ \omega(t-\gamma) \end{bmatrix}^T \begin{bmatrix} -M & M \\ M & -M \end{bmatrix} \begin{bmatrix} \omega(t) \\ \omega(t-\gamma) \end{bmatrix}.$$

Lemma 2.2 ([Boyd et al., 1994] (Schur Complement)). Given constant matrices $\Omega_1$, $\Omega_2$ and $\Omega_3$ with appropriate dimensions, where $\Omega_1^T = \Omega_1$ and $\Omega_2 = \Omega_2 > 0$, then

$$\Omega_1 + \Omega_2^T \Omega_2^{-1} \Omega_3 < 0,$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_2^T \\ * & -\Omega_2 \end{bmatrix} < 0,$$

or

$$\begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

3. State estimation for genetic regulatory networks

In this section, to discuss asymptotic stability for the error system (5), we first consider a delay-dependent condition such that the resulting estimation error system is asymptotically stable. The design of an appropriate state estimator can be achieved by solving a corresponding set of LMI.

Theorem 3.1. For given positive scalars $\overline{\tau}$, $\sigma$, $\rho$ ($0 < \rho < 1$), $\mu_1$ and $\mu_2$, the system (5) is globally asymptotically stable, if there exist symmetric positive definite matrices $P_1, P_2, Q_0, Z_0(\alpha = 1, \ldots, 6)$, any matrices $U_1, U_2, X_1, X_2$ and positive diagonal matrices $W$ with appropriate dimensions, such that the following LMIs hold:

$$Z_1 + (1 - \mu_2)Z_3 > 0, \ Z_2 + (1 - \mu_2)Z_3 > 0, \ Z_4 + (1 - \mu_1)Z_6 > 0, \ Z_5 + (1 - \mu_1)Z_6 > 0,$$

$$\begin{bmatrix} \Pi_{1,1} & 0 & \Pi_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \Pi_{2,2} & U_2D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Pi_{3,3} & \frac{1}{\rho \overline{\tau}}Z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{4,4} & \frac{1}{(1 - \rho \overline{\tau})^2}Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Pi_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{6,6} & \frac{1}{\rho \overline{\tau}}Z_4 & 0 & 0 & 0 & LW \\ * & * & * & * & * & * & \Pi_{7,7} & \frac{1}{(1 - \rho \overline{\tau})^2}Z_5 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Pi_{8,8} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Pi_{9,9} & 0 & U_1B \\ * & * & * & * & * & * & * & * & * & \Pi_{10,10} & 0 \\ * & * & * & * & * & * & * & * & * & * & -W \end{bmatrix} < 0.$$
\[
\Pi_2 = \begin{pmatrix} 
\Pi_{1,1} & 0 & \Pi_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Pi_{1,9} & 0 & U_1 B \\
* & \Pi_{2,2} & U_2 D & 0 & 0 & 0 & 0 & \Pi_{2,7}^{(1)} & 0 & 0 & \Pi_{2,10} & 0 \\
* & * & \Pi_{3,3} & 1 & \frac{1}{\rho_3^2} Z_1 & 0 & 0 & 0 & 0 & 0 & \rho^2 U_2^T U_2^T & 0 \\
* & * & * & \Pi_{4,4} & \frac{1}{(1-\rho_3^2)} Z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Pi_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Pi_{6,6}^{(1)} & \Pi_{6,7}^{(1)} & \Pi_{6,8}^{(1)} & 0 & 0 & LW & 0 \\
* & * & * & * & * & \Pi_{7,7}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Pi_{8,8} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Pi_{9,9} & 0 & U_1 B & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Pi_{10,10}^{(1)} & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & -W 
\end{pmatrix}< 0. \\
(8)
\]

\[
\Pi_3 = \begin{pmatrix} 
\Pi_{1,1} & 0 & 0 & \Pi_{1,4}^{(2)} & 0 & 0 & 0 & 0 & \Pi_{1,9} & 0 & U_1 B \\
* & \Pi_{2,2} & U_2 D & 0 & 0 & \Pi_{2,6} & 0 & 0 & 0 & 0 & \Pi_{2,10} & 0 \\
* & * & \Pi_{3,3}^{(2)} & \Pi_{3,4}^{(2)} & \Pi_{3,5}^{(2)} & 0 & 0 & 0 & 0 & 0 & \rho^2 U_2^T U_2^T & 0 \\
* & * & \Pi_{4,4}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Pi_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Pi_{6,6} & \frac{1}{\rho_3^2} Z_4 & 0 & 0 & 0 & 0 & LW & 0 \\
* & * & * & * & * & \Pi_{7,7} & \frac{1}{(1-\rho_3^2)} Z_4 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Pi_{8,8} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Pi_{9,9}^{(2)} & 0 & U_1 B & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Pi_{10,10}^{(1)} & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & -W 
\end{pmatrix}< 0. \\
(9)
\]

\[
\Pi_4 = \begin{pmatrix} 
\Pi_{1,1} & 0 & 0 & \Pi_{1,4}^{(2)} & 0 & 0 & 0 & 0 & \Pi_{1,9} & 0 & U_1 B \\
* & \Pi_{2,2} & U_2 D & 0 & 0 & \Pi_{2,6} & 0 & 0 & \Pi_{2,7}^{(1)} & 0 & 0 & \Pi_{2,10} & 0 \\
* & * & \Pi_{3,3}^{(2)} & \Pi_{3,4}^{(2)} & \Pi_{3,5}^{(2)} & 0 & 0 & 0 & 0 & 0 & \rho^2 U_2^T U_2^T & 0 \\
* & * & \Pi_{4,4}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Pi_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Pi_{6,6}^{(1)} & \Pi_{6,7}^{(1)} & \Pi_{6,8}^{(1)} & 0 & 0 & LW & 0 \\
* & * & * & * & * & \Pi_{7,7}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Pi_{8,8} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Pi_{9,9}^{(2)} & 0 & U_1 B & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Pi_{10,10}^{(1)} & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & -W 
\end{pmatrix}< 0. \\
(10)
\]
Moreover, the estimator gain matrices are given by $K_1 = U_1^{-1}X_1$ and $K_2 = U_2^{-1}X_2$.

**Proof.** Consider the Lyapunov–Krasovskii functional

$$V(t) = V_1(t) + V_2(t) + V_3(t),$$

where

$$V_1(t) = \dot{X}(t)^T P_1 \dot{X}(t) + \dot{Y}(t)^T P_2 \dot{Y}(t)$$

$$V_2(t) = \int_{t-T}^{t} \dot{X}(s)^T P_1 \dot{X}(s) ds + \int_{t-T}^{t} \dot{Y}(s)^T P_2 \dot{Y}(s) ds + \int_{t-T}^{t} \dot{X}(s)^T P_1 \dot{X}(s) ds + \int_{t-T}^{t} \dot{Y}(s)^T P_2 \dot{Y}(s) ds$$

$$V_3(t) = \int_{t-T}^{t} \dot{X}(s)^T P_1 \dot{X}(s) ds + \int_{t-T}^{t} \dot{Y}(s)^T P_2 \dot{Y}(s) ds + \int_{t-T}^{t} \dot{X}(s)^T P_1 \dot{X}(s) ds + \int_{t-T}^{t} \dot{Y}(s)^T P_2 \dot{Y}(s) ds$$

Now, calculating the derivative of $V(t)$ along the trajectory of the system (5) gives

$$\dot{V}(t) \leq V_1(t) + V_2(t) + V_3(t)$$
where
\[ V_1(t) = 2X^T(t)P_1\dot{X}(t) + 2Y^T(t)P_2\dot{Y}(t), \]
(13)
\[ V_2(t) \leq X^T(t)Q_0X(t) - X^T(t - \rho\sigma)Q_0X(t - \rho\sigma) + Y^T(t - \rho\sigma)Q_0Y(t - \rho\sigma) - Y^T(t - \rho\sigma)Q_0Y(t - \rho\sigma) + Y^T(t)Q_0Y(t) - \frac{3}{2}(t)Q_0(t), \]
(14)
\[ V_3(t) \leq X^T(t)((\rho\sigma)Z_1 + (1 - \rho)Z_2 + \sigma(t)Z_2)\dot{X}(t) + Y^T(t)((\rho\sigma)Z_4 + (1 - \rho)Z_5 + \tau(t)Z_6)\dot{Y}(t). \]
(15)
Here, we have considered the last six terms of (15) for estimating the upper bounds as follows:

**Case 1:** If 0 \leq \sigma(t) \leq \rho\sigma and 0 \leq \tau(t) \leq \rho\tau, we get
\[ -\int_{t-\rho\sigma}^{t} X^T(s)Z_1(s)ds - \int_{t-\rho\sigma}^{t} X^T(s)Z_2(s)ds - \int_{t-\rho\sigma}^{t} Y^T(s)Z_4(s)ds - \int_{t-\rho\sigma}^{t} Y^T(s)Z_5(s)ds. \]
(16)
Note that (Z_1 + (1 - \mu_2)Z_3) > 0, (Z_4 + (1 - \mu_1)Z_6) > 0 and using **Lemma 2.1**, it follows that
\[ -\int_{t-\rho\sigma}^{t} X^T(s)Z_1(s)ds \leq \begin{bmatrix} \mathbb{R}(t) \\ \mathbb{R}(t - \sigma(t)) \end{bmatrix}^T \begin{bmatrix} (Z_1 + (1 - \mu_2)Z_3) \\ (Z_4 + (1 - \mu_1)Z_6) \end{bmatrix}, \]
(17)
\[ -\int_{t-\rho\sigma}^{t} X^T(s)Z_2(s)ds \leq \begin{bmatrix} \mathbb{R}(t) \\ \mathbb{R}(t - \sigma(t)) \end{bmatrix}^T \begin{bmatrix} Z_1 \\ Z_4 \end{bmatrix}, \]
(18)
\[ -\int_{t-\rho\sigma}^{t} Y^T(s)Z_4(s)ds \leq \begin{bmatrix} \mathbb{R}(t - \rho\sigma) \\ \mathbb{R}(t - \sigma(t)) \end{bmatrix}^T \begin{bmatrix} Z_2 \\ Z_4 \end{bmatrix}, \]
(19)
\[ -\int_{t-\rho\sigma}^{t} Y^T(s)Z_5(s)ds \leq \begin{bmatrix} \mathbb{R}(t - \rho\sigma) \\ \mathbb{R}(t - \sigma(t)) \end{bmatrix}^T \begin{bmatrix} Z_2 \\ Z_4 \end{bmatrix}, \]
(20)
\[ -\int_{t-\rho\tau}^{t} Y^T(s)Z_6(s)ds \leq \begin{bmatrix} \mathbb{R}(t - \rho\tau) \\ \mathbb{R}(t - \tau(t)) \end{bmatrix}^T \begin{bmatrix} Z_5 \\ Z_5 \end{bmatrix}, \]
(21)
\[ -\int_{t-\rho\tau}^{t} Y^T(s)Z_7(s)ds \leq \begin{bmatrix} \mathbb{R}(t - \rho\tau) \\ \mathbb{R}(t - \tau(t)) \end{bmatrix}^T \begin{bmatrix} Z_5 \\ Z_5 \end{bmatrix}, \]
(22)
It is obvious from (2) that
\[ 2Y^T(t)Wg(\mathbb{R}(t - \tau(t))) - 2g^T(\mathbb{R}(t - \tau(t)))Wg(\mathbb{R}(t - \tau(t))) \geq 0, \]
(23)
holds for any positive diagonal matrix W. On other hand, based on Eq. (5), one has for any matrices U_1 and U_2 such that
\[ 0 = 2\mathbb{R}(t) + \dot{X}(t)U_1 - [A + K]M\mathbb{R}(t) + Bg(\mathbb{R}(t - \tau(t)) - \dot{R}(t)), \]
(24)
Substituting (17)–(22) and adding (23)–(25) into (12), we have
\begin{equation}
V(t) \leq \xi^2(t)(\Pi_1)\xi(t)
\end{equation}
where
\begin{equation}
\xi(t) = (\xi^2(t), \xi^3(t), \xi^4(t), \xi^5(t) - \rho \xi^2(t), \xi^6(t) - \rho \xi^2(t), \xi^7(t) - \rho \xi^2(t), \xi^8(t), \xi^9(t))
\end{equation}
From (7), it is easy to see that \( V(t) < 0 \) holds if \( \Pi_1 < 0 \). \( 0 \leq \sigma(t) \leq \rho \sigma \) and \( 0 \leq \tau(t) \leq \rho \tau \).

Case 2: If \( 0 \leq \sigma(t) \leq \rho \sigma \) and \( \rho \tau \leq \tau(t) \leq T \), we get
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^2(s)Z_1(s)ds - \int_{t-\rho \tau}^{t} \xi^2(s)Z_2(s)ds - (1 - \mu_2) \int_{t-\rho \tau}^{t} \xi^2(s)Z_3(s)ds - \int_{t-\rho \tau}^{t} \xi^2(s)Z_4(s)ds
\end{equation}
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^3(s)Z_5(s)ds - \int_{t-\rho \tau}^{t-\rho \tau} \xi^3(s)Z_6(s)ds = \int_{t-\rho \tau}^{t} \xi^3(s)Z_1(s) + \int_{t-\rho \tau}^{t} \xi^3(s)Z_2(s)ds - \int_{t-\rho \tau}^{t} \xi^3(s)Z_3(s)ds
\end{equation}
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^4(s)Z_4(s)ds - \int_{t-\rho \tau}^{t} \xi^5(s)Z_5(s)ds - \int_{t-\rho \tau}^{t} \xi^5(s)Z_6(s)ds - \int_{t-\rho \tau}^{t} \xi^6(s)Z_7(s)ds.
\end{equation}
Noticing that \( Z_2 + (1 - \mu_2)Z_3 > 0 \), \( Z_2 + (1 - \mu_1)Z_4 > 0 \), \( Z_2 + (1 - \mu_1)Z_6 > 0 \) and using similar approach in Case 1, we get
\begin{equation}
V(t) \leq \xi^2(t)(\Pi_2)\xi(t).
\end{equation}
From (8), it is easy to see that \( V(t) < 0 \) holds if \( \Pi_2 < 0 \), \( 0 \leq \sigma(t) \leq \rho \sigma \) and \( \rho \tau \leq \tau(t) \leq T \).

Case 3: If \( \rho \sigma \leq \sigma(t) \leq \sigma \) and \( 0 \leq \tau(t) \leq \rho \tau \), we get
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^2(s)Z_1(s)ds - \int_{t-\rho \tau}^{t} \xi^2(s)Z_2(s)ds - (1 - \mu_2) \int_{t-\rho \tau}^{t} \xi^2(s)Z_3(s)ds - \int_{t-\rho \tau}^{t} \xi^2(s)Z_4(s)ds
\end{equation}
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^3(s)Z_5(s)ds - \int_{t-\rho \tau}^{t-\rho \tau} \xi^3(s)Z_6(s)ds = \int_{t-\rho \tau}^{t} \xi^3(s)Z_1(s) + \int_{t-\rho \tau}^{t} \xi^3(s)Z_2(s)ds - \int_{t-\rho \tau}^{t} \xi^3(s)Z_3(s)ds
\end{equation}
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^4(s)Z_4(s)ds - \int_{t-\rho \tau}^{t} \xi^5(s)Z_5(s)ds - \int_{t-\rho \tau}^{t} \xi^5(s)Z_6(s)ds - \int_{t-\rho \tau}^{t} \xi^6(s)Z_7(s)ds.
\end{equation}
Noticing that \( Z_2 + (1 - \mu_2)Z_3 > 0 \), \( Z_2 + (1 - \mu_1)Z_4 > 0 \), \( Z_2 + (1 - \mu_1)Z_6 > 0 \) and using similar approach in Case 1, we have
\begin{equation}
V(t) \leq \xi^2(t)(\Pi_3)\xi(t).
\end{equation}
Then, from (9), it is easy to see that \( V(t) < 0 \) holds if \( \Pi_3 < 0 \), \( \rho \sigma \leq \sigma(t) \leq \sigma \) and \( 0 \leq \tau(t) \leq \rho \tau \).

Case 4: If \( \rho \sigma \leq \sigma(t) \leq \sigma \) and \( \rho \tau \leq \tau(t) \leq T \), we get
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^2(s)Z_1(s)ds - \int_{t-\rho \tau}^{t} \xi^2(s)Z_2(s)ds - (1 - \mu_2) \int_{t-\rho \tau}^{t} \xi^2(s)Z_3(s)ds - \int_{t-\rho \tau}^{t} \xi^2(s)Z_4(s)ds
\end{equation}
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^3(s)Z_5(s)ds - \int_{t-\rho \tau}^{t-\rho \tau} \xi^3(s)Z_6(s)ds = \int_{t-\rho \tau}^{t} \xi^3(s)Z_1(s) + \int_{t-\rho \tau}^{t} \xi^3(s)Z_2(s)ds - \int_{t-\rho \tau}^{t} \xi^3(s)Z_3(s)ds
\end{equation}
\begin{equation}
- \int_{t-\rho \tau}^{t} \xi^4(s)Z_4(s)ds - \int_{t-\rho \tau}^{t} \xi^5(s)Z_5(s)ds - \int_{t-\rho \tau}^{t} \xi^5(s)Z_6(s)ds - \int_{t-\rho \tau}^{t} \xi^6(s)Z_7(s)ds.
\end{equation}
By the fact \( Z_2 + (1 - \mu_2)Z_3 > 0 \), \( Z_2 + (1 - \mu_2)Z_4 > 0 \), \( Z_2 + (1 - \mu_1)Z_6 > 0 \) and the approach in Case 1, finally we get
\begin{equation}
V(t) \leq \xi^2(t)(\Pi_4)\xi(t).
\end{equation}
From (10), it is easy to see that \( V(t) < 0 \) holds if \( \Pi_4 < 0 \), \( \rho \sigma \leq \sigma(t) \leq \sigma \) and \( \rho \tau \leq \tau(t) \leq T \). From LMIs (6)–(10), the error system (5) is globally asymptotically stable. Thus, the proof is completed. □

**Remark 3.1.** Theorem 3.1 provide a new delay-dependent stability criterion by using tuning parameter \( \rho(0 < \rho < 1) \). The intervals \([0, \sigma]\) and \([0, T]\) are divided into four subintervals such as \([0, \rho \sigma]\) \([\rho \sigma, \sigma]\), \([0, \rho \tau]\) and \([\rho \tau, T]\) in which the information of delayed estimation error states \( \xi(t - \rho \sigma) \) and \( \xi(t - \rho \tau) \) can be taken into account.
4. State estimation for genetic regulatory networks with randomly occurring uncertainties

As discussed in introduction, while modeling a dynamical system, one can hardly obtain the exact model because of modeling errors, external perturbation and parameter fluctuations. So it is important to study the robust problem of such networks with uncertainties. In the following, we will shift the equilibrium point $(\bar{x}, \bar{y})$ of the system (1) to the origin by letting $x(t) = \bar{x}(t) - \bar{x}$ and $y(t) = \bar{y}(t) - \bar{y}$. Hence, system (1) can be transformed into the following form in Liang et al. (2009), Liang and Lam (2010) and Lv et al. (2011).

$$\begin{align*}
\dot{x}(t) &= -Ax(t) + Bf(y(t) - \tau(t)), \\
\dot{y}(t) &= -Cy(t) + Dx(t - \sigma(t)),
\end{align*}$$

(30)

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, $y(t) = [y_1(t), y_2(t), \ldots, y_m(t)]^T$; the nonlinear function $f(y(t)) = [f_1(y_1(t)), f_2(y_2(t)), \ldots, f_m(y_m(t))]^T$ with $f'(y(t)) = g(y(t) + \bar{y}) - g(\bar{y})$.

From (2), it satisfies the following condition

$$f(y)f(y) - \Delta y \leq 0.$$  (31)

As a consequence, in this section, we will focus on the following delayed uncertain GRNs with parameter uncertainties:

$$\begin{align*}
\dot{x}(t) &= -(A + \alpha(t)\Delta A(t))x(t) + (B + \beta(t)\Delta B(t))f(y(t) - \tau(t)), \\
\dot{y}(t) &= -(C + \gamma(t)\Delta C(t))y(t) + (D + \eta(t)\Delta D(t))x(t - \sigma(t)),
\end{align*}$$

(32)

with the measurements outputs

$$\begin{align*}
\tilde{x}_1(t) &= (M + \upsilon(t)\Delta M(t))x(t), \\
\tilde{x}_2(t) &= (N + \lambda(t)\Delta N(t))y(t),
\end{align*}$$

(33)

where the stochastic variables $\alpha(t), \beta(t), \gamma(t), \eta(t), \upsilon(t)$ and $\lambda(t)$ are randomly occurring uncertainties; $\Delta A(t), \Delta B(t), \Delta C(t), \Delta D(t), \Delta M(t)$ and $\Delta N(t)$ are the system uncertainties satisfying

$$\begin{align*}
[\Delta A(t) &\quad \Delta B(t)] = H_1F(t)[E_1 \quad E_2], \\
[\Delta C(t) &\quad \Delta D(t)] = H_2F(t)[E_3 \quad E_4], \\
\Delta M(t) &= H_3F(t)E_5, \\
\Delta N(t) &= H_4F(t)E_6.
\end{align*}$$

(34)

Here, $H_1, H_2, H_3, H_4, E, \{1, 2, \ldots, 6\}$ are known real constant matrices, and $F(t)$ denotes unknown time-varying matrix functions satisfying $F^T(t)F(t) \leq I$.  (35)

To account for the phenomena of randomly occurring uncertainties, natural assumptions on the stochastic variables $\alpha(t), \beta(t), \gamma(t), \eta(t), \upsilon(t)$, and $\lambda(t)$, which are mutually independent Bernoulli-distributed white sequences, are as follows:

$$\begin{align*}
Pr[\alpha(t) = 1] &= \alpha, & Pr[\alpha(t) = 0] &= 1 - \alpha, \\
Pr[\beta(t) = 1] &= \beta, & Pr[\beta(t) = 0] &= 1 - \beta, \\
Pr[\gamma(t) = 1] &= \gamma, & Pr[\gamma(t) = 0] &= 1 - \gamma, \\
Pr[\eta(t) = 1] &= \eta, & Pr[\eta(t) = 0] &= 1 - \eta, \\
Pr[\upsilon(t) = 1] &= \upsilon, & Pr[\upsilon(t) = 0] &= 1 - \upsilon, \\
Pr[\lambda(t) = 1] &= \lambda, & Pr[\lambda(t) = 0] &= 1 - \lambda,
\end{align*}$$

where $\alpha \in [0, 1], \beta \in [0, 1], \gamma \in [0, 1], \eta \in [0, 1], \upsilon \in [0, 1]$ and $\lambda \in [0, 1]$ are known constants.

Now, we have considered the corresponding state estimate of the concentrations of mRNA and protein in (32) from the available network outputs in (33) is of the following form:

$$\begin{align*}
\dot{\hat{x}}(t) &= -A\hat{x}(t) + Bf(\hat{y}(t - \tau(t))) + K_1[x(t) - M\hat{x}(t)], \\
\dot{\hat{y}}(t) &= -C\hat{y}(t) + D\hat{x}(t - \sigma(t)) + K_2[y(t) - N\hat{y}(t)],
\end{align*}$$

(36)

where $\hat{x}(t)$ and $\hat{y}(t) \in R^n$ are the estimations of $x(t)$ and $y(t)$, respectively.

Define the error vector by $\check{x}(t) = x(t) - \hat{x}(t)$, $\check{y}(t) = y(t) - \hat{y}(t)$. Then, the error dynamics can be directly obtained from (32) and (36):

$$\begin{align*}
\dot{\check{x}}(t) &= -(A + K_1M)\check{x}(t) - [\alpha(t)\Delta A(t)]\check{x}(t) + Bf(\check{y}(t - \tau(t))) + \beta(t)\Delta B(t)f(\check{y}(t - \tau(t))), \\
\dot{\check{y}}(t) &= -(C + K_2N)\check{y}(t) - [\gamma(t)\Delta C(t)]\check{y}(t) + (D + \eta(t)\Delta D(t))\check{x}(t - \sigma(t)).
\end{align*}$$

(37)

Considering (32) and (37) together, the following augmented system can be obtained

$$\begin{align*}
\dot{\hat{x}}(t) &= -(A + \alpha(t)\Delta A(t))\hat{x}(t) + (B + \beta(t)\Delta B(t))f(y(t - \tau(t))), \\
\dot{\check{x}}(t) &= -(A + K_1M)\check{x}(t) - [\alpha(t)\Delta A(t)]\check{x}(t) + Bf(\check{y}(t - \tau(t))) + \beta(t)\Delta B(t)f(\check{y}(t - \tau(t))), \\
\dot{\hat{y}}(t) &= -(C + \gamma(t)\Delta C(t))\hat{y}(t) + (D + \eta(t)\Delta D(t))\check{x}(t - \sigma(t)), \\
\dot{\check{y}}(t) &= -(C + K_2N)\check{y}(t) - [\gamma(t)\Delta C(t)]\check{y}(t) + (D + \eta(t)\Delta D(t))\check{x}(t - \sigma(t)).
\end{align*}$$

(38)

Before the design of a state estimator for GRNs (38), the following lemma should be remembered.

**Lemma 4.1 (Wang et al., 2006).** Let $M = M^T$, $H$ and $E$ be real matrices of appropriate dimensions $F^T(t)F(t) \leq I$ then

$$M + HF(t)E + E^TF^T(t)H^T < 0,$$

where $\alpha \in [0, 1], \beta \in [0, 1], \gamma \in [0, 1], \eta \in [0, 1], \upsilon \in [0, 1]$ and $\lambda \in [0, 1]$ are known constants.
if and only if there exists a positive scalar $\epsilon > 0$ such that

$$M + \frac{1}{\epsilon}HH^T + \epsilon E^TE < 0.$$ 

Now, we have the following result on state estimation problem of GRNs with ROUs and delays.

**Theorem 4.1.** For given positive scalars $\overline{\Sigma}, \overline{\mu}, \beta (0 < \mu < 1), \mu_1$ and $\mu_2$, the system (38) is robustly stochastically asymptotically stable, if there exist symmetric positive definite matrices $P_1, P_2, Q_a, \tildes_2, \tildes_3, W$ any matrices $U_a (a = 1, \ldots, 6), X_1, X_2$, positive diagonal matrices $W_1, W_2$ with appropriate dimensions and scalar $\epsilon_i > 0 (i = 1, \ldots, 6)$, such that the following LMIs hold:

$$\mathcal{Z}_1 = \begin{bmatrix} \Omega_1 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ \ast & \ast & -\epsilon_2 I & 0 & 0 & 0 & 0 \\ \ast & \ast & \ast & -\epsilon_3 I & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & -\epsilon_4 I & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & -\epsilon_5 I & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast & -\epsilon_6 I \end{bmatrix} < 0,$$

(39)

$$\mathcal{Z}_2 = \begin{bmatrix} \Omega_2 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ \ast & \ast & -\epsilon_2 I & 0 & 0 & 0 & 0 \\ \ast & \ast & \ast & -\epsilon_3 I & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & -\epsilon_4 I & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & -\epsilon_5 I & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast & -\epsilon_6 I \end{bmatrix} < 0,$$

(40)

$$\mathcal{Z}_3 = \begin{bmatrix} \Omega_3 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ \ast & \ast & -\epsilon_2 I & 0 & 0 & 0 & 0 \\ \ast & \ast & \ast & -\epsilon_3 I & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & -\epsilon_4 I & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & -\epsilon_5 I & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast & -\epsilon_6 I \end{bmatrix} < 0,$$

(41)

$$\mathcal{Z}_4 = \begin{bmatrix} \Omega_4 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ & -\epsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ \ast & \ast & -\epsilon_2 I & 0 & 0 & 0 & 0 \\ \ast & \ast & \ast & -\epsilon_3 I & 0 & 0 & 0 \\ \ast & \ast & \ast & \ast & -\epsilon_4 I & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & -\epsilon_5 I & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast & -\epsilon_6 I \end{bmatrix} < 0,$$

(42)

and

$$\dot{\tildes}_1 + (1 - \mu_2)\tildes_2 > 0, \quad \dot{\tildes}_2 + (1 - \mu_2)\tildes_3 > 0, \quad \dot{\tildes}_4 + (1 - \mu_1)\tildes_5 > 0, \quad \dot{\tildes}_5 + (1 - \mu_1)\tildes_6 > 0.$$  

(43)

Then, the estimator gain matrices can be designed as $K_1 = U_5^{-1}X_1$ and $K_2 = U_6^{-1}X_2$. All the entries in the matrices (39)-(43) are given in Appendix A.

**Proof.** Consider the Lyapunov–Krasovskii functional

$$\dot{\mathcal{V}}(t) = \dot{\mathcal{V}}_1(t) + \dot{\mathcal{V}}_2(t) + \dot{\mathcal{V}}_3(t),$$

(44)
where

\[
\dot{V}_1(t) = e_1^T(t)\ddot{p}_1 e_1(t) + e_2^T(t)\ddot{p}_2 e_2(t),
\]

\[
\dot{V}_2(t) = \int_{-\pi}^{\pi} e_1^T(s)\dot{Q}_1 e_1(s)ds + \int_{-\pi}^{\pi} e_2^T(s)\dot{Q}_2 e_2(s)ds + \int_{-\pi}^{\pi} e_1^T(s)\dot{Q}_3 e_1(s)ds + \int_{-\pi}^{\pi} e_2^T(s)\dot{Q}_4 e_2(s)ds + \int_{-\pi}^{\pi} e_1^T(s)\dot{Q}_5 e_1(s)ds + \int_{-\pi}^{\pi} e_2^T(s)\dot{Q}_6 e_2(s)ds
\]

\[
\dot{V}_3(t) = \int_{0}^{\pi} \int_{\pi}^{\theta} e_1^T(s)\dot{\alpha}_1 e_1(s)dsd\theta + \int_{-\pi}^{\pi} e_2^T(s)\dot{\alpha}_2 e_2(s)dsd\theta + \int_{-\pi}^{\pi} e_1^T(s)\dot{\alpha}_3 e_1(s)dsd\theta + \int_{-\pi}^{\pi} e_2^T(s)\dot{\alpha}_4 e_2(s)dsd\theta
\]

where \(e_s = [x(t) \dot{x}(t)]^T\) and \(e_y = [y(t) \dot{y}(t)]^T\).

From (31), the following inequality holds for any positive diagonal matrix \(W_1\) and \(W_2\).

\[
2y^T(t - \tau(t))W_1 f(y(t - \tau(t))) - 2y^T(t)\dot{W}_1 f(y(t - \tau(t))) \geq 0.
\]

\[
2\dot{y}^T(t - \tau(t))W_2 f(\dot{y}(t - \tau(t))) - 2\dot{y}^T(t)\ddot{W}_2 f(\dot{y}(t - \tau(t))) \geq 0.
\]

From the equations (32) and (37), the following equations also holds for \(U_1, U_2, U_3, U_4, U_5\) and \(U_6\).

\[
0 = \mathbb{E}(2[x(t)U_1 + x(t)U_2]) - \mathbb{E}[\alpha(t)\Delta(t)x(t) + \beta(t)\Delta(t)\dot{x}(t) - \dot{x}(t)].
\]

\[
0 = \mathbb{E}(2[y(t)U_3 + y(t)U_4]) - \mathbb{E}[(C + \gamma(t)\Delta(t))y(t) + \delta(t)\Delta(t)x(t) - \dot{x}(t)].
\]

\[
0 = \mathbb{E}(2[x(t) + \tilde{x}(t)]U_5) - \mathbb{E}[\alpha(t)\Delta(t) + K_1u(t)\Delta(t)]x(t) + \mathbb{E}[\beta(t)\Delta(t)f(y(t - \tau(t))) - \dot{x}(t)].
\]

\[
0 = \mathbb{E}(2[y(t) + \tilde{y}(t)]U_6) - \mathbb{E}[\gamma(t)\Delta(t) + K_2u(t)\Delta(t)]y(t) + \mathbb{E}[\delta(t)\Delta(t)x(t) - \dot{x}(t)].
\]

Note that \(L\) is the infinitesimal operator of \(\tilde{V}(t)\) as in Shen et al. (2011). Then, taking the time derivative of \(\bar{V}(t)\) and using similar method discussed in Case 1 of Theorem 3.1 gives that

\[
\mathbb{E}[L\bar{V}(t)] \leq \mathbb{E}[L\tilde{V}(t) + \Delta\Omega(t)]\tilde{e}_1(t) < 0,
\]

where

\[
\tilde{e}_1(t) = (x(t) \dot{x}(t) \gamma^T(t) y(t) \dot{y}(t)) \gamma(t) - \gamma(t) \gamma(t) - \gamma(t) - \gamma(t) - \gamma(t) - \gamma(t) - \gamma(t) - \gamma(t).
\]

\[
\Delta\Omega(t) = e_1^T(t)\gamma(t) \gamma(t) e_1(t) + e_1^T(t)\gamma(t) \gamma(t) e_1(t) + e_2^T(t)\gamma(t) \gamma(t) e_2(t) + e_2^T(t)\gamma(t) \gamma(t) e_2(t) + e_3^T(t)\gamma(t) \gamma(t) e_3(t) + e_3^T(t)\gamma(t) \gamma(t) e_3(t) + e_4^T(t)\gamma(t) \gamma(t) e_4(t) + e_4^T(t)\gamma(t) \gamma(t) e_4(t) + e_5^T(t)\gamma(t) \gamma(t) e_5(t) + e_5^T(t)\gamma(t) \gamma(t) e_5(t) + e_6^T(t)\gamma(t) \gamma(t) e_6(t) + e_6^T(t)\gamma(t) \gamma(t) e_6(t).
\]

By Lemma 4.1, we have

\[
\Delta\Omega(t) \leq e_1^T(t)\gamma(t) \gamma(t) e_1(t) + e_1^T(t)\gamma(t) \gamma(t) e_1(t) + e_2^T(t)\gamma(t) \gamma(t) e_2(t) + e_2^T(t)\gamma(t) \gamma(t) e_2(t) + e_3^T(t)\gamma(t) \gamma(t) e_3(t) + e_3^T(t)\gamma(t) \gamma(t) e_3(t) + e_4^T(t)\gamma(t) \gamma(t) e_4(t) + e_4^T(t)\gamma(t) \gamma(t) e_4(t) + e_5^T(t)\gamma(t) \gamma(t) e_5(t) + e_5^T(t)\gamma(t) \gamma(t) e_5(t) + e_6^T(t)\gamma(t) \gamma(t) e_6(t) + e_6^T(t)\gamma(t) \gamma(t) e_6(t) + e_7^T(t)\gamma(t) \gamma(t) e_7(t) + e_7^T(t)\gamma(t) \gamma(t) e_7(t) + e_8^T(t)\gamma(t) \gamma(t) e_8(t) + e_8^T(t)\gamma(t) \gamma(t) e_8(t) + e_9^T(t)\gamma(t) \gamma(t) e_9(t) + e_9^T(t)\gamma(t) \gamma(t) e_9(t) + e_10^T(t)\gamma(t) \gamma(t) e_10(t) + e_10^T(t)\gamma(t) \gamma(t) e_10(t) + e_11^T(t)\gamma(t) \gamma(t) e_11(t) + e_11^T(t)\gamma(t) \gamma(t) e_11(t).
\]

By Lemma 2.2, we know that Eq. (39) is equivalent to the following inequality

\[
\Omega_1 + e_1^T(t)\gamma(t) \gamma(t) e_1(t) + e_2^T(t)\gamma(t) \gamma(t) e_2(t) + e_3^T(t)\gamma(t) \gamma(t) e_3(t) + e_4^T(t)\gamma(t) \gamma(t) e_4(t) + e_5^T(t)\gamma(t) \gamma(t) e_5(t) + e_6^T(t)\gamma(t) \gamma(t) e_6(t) + e_7^T(t)\gamma(t) \gamma(t) e_7(t) + e_8^T(t)\gamma(t) \gamma(t) e_8(t) + e_9^T(t)\gamma(t) \gamma(t) e_9(t) + e_10^T(t)\gamma(t) \gamma(t) e_10(t) + e_11^T(t)\gamma(t) \gamma(t) e_11(t)
\]

By Table 1. We gain matrices \(K_1\) and \(K_2\) for different value of \(\gamma, \rho, \mu_1, \mu_2, \alpha\) = 0.2.

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<th>(K_2 )</th>
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<td>0.3155</td>
<td>0.8695</td>
<td>1.0458</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8</td>
<td>1.5</td>
<td>0.5708</td>
<td>0.1524</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.2723</td>
<td>0.4584</td>
</tr>
<tr>
<td></td>
<td>0.0293</td>
<td>0.3169</td>
<td>0.1225</td>
<td>0.2059</td>
</tr>
</tbody>
</table>
where
\[
\begin{align*}
\psi_1 &= [-\alpha E_1 \quad 0_{1 \times 19} \quad \beta E_2 \quad 0], \\
\psi_2 &= [0 \quad 0 \quad -\gamma E_3 \quad 0 \quad \eta E_4 \quad 0_{1 \times 17}], \\
\psi_3 &= [-\nu E_5 \quad 0_{1 \times 21}], \\
\psi_4 &= [0 \quad 0 \quad -\lambda E_6 \quad 0_{1 \times 19}],
\end{align*}
\]
which ensures \(\Omega_1 + \Delta \Omega_2(t) < 0\). Therefore, the augmented system (38) is robustly stochastically asymptotically stable for the state estimator of uncertain GRN (32) with measurement (33) if \(\tilde{E}_2 < 0\), \(0 \leq \sigma(t) \leq \rho \sigma\) and \(0 \leq \tau(t) \leq \rho \tau\). We can easily prove the remaining Cases by the similar approach of Theorem 3.1. The proof is completed. \(\Box\)

**Remark 4.1.** Recently, the problem of robust state estimation for stochastic genetic regulatory networks have been proposed in Liang and Lam (2010). Further, Lv et al. (2011) and Liang et al. (2009) have investigated the state estimation for Markov-type genetic regulatory networks with delays. In these works, all the delay dependent stability criteria are derived in term of LMIs. Different from existing literature, in this paper, the norm-bounded uncertainties enter into the GRNs in randomly ways, and such ROUs obey certain mutually uncorrelated Bernoulli distributed white noise sequences. Based on appropriate Lyapunov functional, choosing different positive-definite matrices in the decomposed intervals by using tuning parameter \(\rho\), and estimating the upper bounds of some cross term more exactly, a new delay-dependent condition is derived in term of LMIs. It should be pointed out that this is the first attempt for state estimation problem conducted on GRNs with ROU.

**Remark 4.2.** It is noted that the merit of this work has been reported in Remarks 3.1 and 4.1. Besides, when the dimension of system matrices increases, the computational complexity of Theorem 4.1 will also increases. Currently, to overcome this aforementioned disadvantage becomes an important research topic of the delay-dependent related problems.

5. Numerical example

In this section, we will give a numerical example showing the effectiveness of established theoretical results.

![Fig. 2. The state trajectories of the system (32), (36) and (37) with different initial conditions.](image-url)
The dynamics of repressilator has been theoretically predicted and experimentally investigated in *Escherichia coli* (Elowitz and Leibler, 2000). The repressilator is a cyclic negative-feedback loop comprising three genes (lacI, tetR, and cl) and their promoters. The kinetics of the system are determined by six coupled first-order differential equations:

\[
\begin{align*}
\dot{x}_i(t) &= -x_i + \frac{\alpha_i}{1 + y_j}, \\
\dot{y}_j(t) &= -\beta_i (y_j - x_i),
\end{align*}
\]

where \(i = \text{lacI, tetR, cl} \); \(j = \text{cl, lacI, tetR} \), respectively. \(x_i\) and \(y_j\) are the concentrations of the three mRNA and repressor-proteins, and \(\beta_i > 0\) denotes the ratio of the protein decay rate to mRNA decay rate. A specific choice of the coefficients \(\alpha_i\) and \(\beta_i\) were presented in the recent literatures. Taking time delays, and random occurring uncertainties, into account, we adjust the above Eq. (45) into the form of (32) with

\[
B = \begin{bmatrix} 0 & 0 & -1.4 \\
-1.4 & 0 & 0 \\
0 & -1.4 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2.6 & 0 & 0 \\
0 & 2.5 & 0 \\
0 & 0 & 2.6 \end{bmatrix}, \quad D = \begin{bmatrix} 0.9 & 0 & 0 \\
0 & 0.9 & 0 \\
0 & 0 & 0.9 \end{bmatrix},
\]

\(A = \text{diag}(2.8, 3.1, 3.2)\) and the nonlinear regulatory function is taken to be \(g(y) = y^{2}/(1 + y^{2})\). That is, \(L = 0.65 I\). In this example, the parameter uncertainties satisfying (34)–(35) are as follows \(R(t) = \text{diag}(\sin t, \cos t, -\sin t)\).

\[
\begin{align*}
H_1 &= \begin{bmatrix} 0.3 & 0 & -0.3 \\
0.3 & 0.3 & 0.3 \\
0.4 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.2 & 0.2 & 0.22 \\
0.3 & 0.4 & -0.4 \\
-0.4 & 0.03 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} -0.4 & 0.2 & 0.2 \\
0 & -0.3 & 0 \\
0 & 0 & -0.3 \end{bmatrix}, \\
E_2 &= \begin{bmatrix} 0 & 0 & 0 \\
2 & 0 & 0 \\
0 & 0.25 & -0.1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} -0.8 & 0.5 & 0.7 \\
-0.5 & 0 & -0.2 \\
-0.4 & -0.3 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} -0.2 & 0.3 & 0.4 \\
0 & 0 & 0.8 \end{bmatrix},
\end{align*}
\]

Fig. 3. The state trajectories of the system (32), (36) and (37) with different initial conditions.
The parameters in the network measurements (33) are assumed as

\[
M = \begin{bmatrix}
0.5 & -0.6 & 0 \\
0.3 & 0.8 & -0.2 \\
0.2 & 0 & -0.4 \\
\end{bmatrix}, \quad
N = \begin{bmatrix}
0.7 & -0.25 & 0.3 \\
0.4 & 0.2 & -0.3 \\
-0.3 & 0.2 & 0 \\
0 & -0.1 & -0.3 \\
\end{bmatrix}, \quad
H_3 = \begin{bmatrix}
0.5 & -0.5 & 0.42 \\
0.2 & 0.2 & 0.33 \\
-0.4 & 0.22 & 0.23 \\
0 & -0.1 & 0.2 \\
0 & -0.1 & 0.2 \\
\end{bmatrix},
\]

\[
H_4 = \begin{bmatrix}
0 & 0.4 & 0.4 \\
0.2 & 0 & -0.4 \\
\end{bmatrix}, \quad
E_5 = \begin{bmatrix}
0.25 & 0.2 & -0.4 \\
-0.3 & 0.2 & 0 \\
\end{bmatrix}, \quad
E_6 = \begin{bmatrix}
0 & -0.1 & 0.2 \\
0 & -0.1 & 0.2 \\
\end{bmatrix}
\]

and \( \alpha = \beta = \gamma = \eta = \upsilon = \lambda = \alpha'. \)

For this example, by applying Theorem 4.1 and using the MATLAB LMI control Toolbox, we found the corresponding gain matrices for different value of \( \tilde{\sigma}, \tilde{\tau}, \mu_1, \mu_2, \alpha' = 0.2 \) and \( \rho \) which are listed in Table 1.

Next, let us consider a nominal case, that is, \( \Delta A(t) = \Delta B(t) = \Delta C(t) = \Delta D(t) = \Delta M(t) = \Delta N(t) = 0. \) For \( \tilde{\sigma} = \tilde{\tau} = 1, \mu_1 = \mu_2 = 0.5, \rho = 0.8, \) by Theorem 3.1, we can obtain the gain matrices as follows

\[
K_1 = \begin{bmatrix}
8.9890 & 6.3883 \\
-3.5032 & 5.0832 \\
-0.8478 & -1.7254 \\
\end{bmatrix}, \quad
K_2 = \begin{bmatrix}
1.4196 & 1.2236 \\
-0.9654 & 1.6253 \\
1.7345 & -2.7911 \\
\end{bmatrix}
\]

For the purpose of simulation, we chose the time-varying delays as \( \sigma(t) = 0.5 + 0.5 \sin t, \tau(t) = 0.5 + 0.5 \cos t, \) which means \( \tilde{\sigma} = 1, \tilde{\tau} = 1, \mu_1 = \mu_2 = 0.5 \) when \( \rho = 0.5. \) Figs. 2 and 3 are numerical illustrations of the trajectories of delayed GRNs (32) and the state estimator system (36) with different initial conditions. In addition, the trajectories of delayed GRNs (1) and the state estimator system (4) with \( j(t) = [\cos 3 t \sin t \cos t]^T \) and different initial conditions are shown in Figs. 4 and 5. Further, the simulation results of \( \alpha(t), \beta(t), \gamma(t), \eta(t), \upsilon(t), \) and \( \lambda(t) \) are shown in Fig. 6. These simulation results confirm the effectiveness of Theorem 3.1 and 4.1 for the state estimator design for delayed GRNs.

Fig. 4. The state trajectories of the system (1), (4) and (5) with different initial conditions.
Fig. 5. The state trajectories of the system (1), (4) and (5) with different initial conditions.

Fig. 6. Simulations of \( \alpha(t) \), \( \beta(t) \), \( \gamma(t) \), \( \eta(t) \), \( \upsilon(t) \), and \( \lambda(t) \).
6. Conclusions

In this paper, we have considered GRNs with time-delay and randomly occurring uncertainties. A state estimator is designed to estimate the gene states through available sensor measurements, such that the error signals between the GRNs and state estimator is globally stochastically stable. Based on constructing appropriate Lyapunov functional and choosing different Lyapunov matrices in the decomposed intervals by using the tuning parameter \( \rho \), new delay-dependent conditions for finding gain matrices of the state estimator are derived. A numerical example is given to illustrate the applicability and effectiveness of the proposed method. On the other hand, the filtering problem has been extensively studied in the control and signal processing communities in present time. Further, stochastic noise has great effect on the dynamic behaviors of gene regulatory networks. Therefore, our future directions will be focused on the filtering problem with stochastic noise which is a challenging issue for GRNs because of the complex structure of these networks.

Appendix A.

\[
\dot{P}_1 = \text{diag}(P_1, P_2), \quad \dot{P}_2 = \text{diag}(P_3, P_4), \quad \dot{Q}_1 = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \quad \dot{Q}_2 = \begin{bmatrix} Q_4 & Q_5 \\ Q_5^T & Q_6 \end{bmatrix},
\]

\[
\dot{Q}_3 = \begin{bmatrix} Q_7 & Q_8 \\ Q_8^T & Q_9 \end{bmatrix}, \quad \dot{Q}_4 = \begin{bmatrix} Q_{10} & Q_{11} \\ Q_{11}^T & Q_{12} \end{bmatrix}, \quad \dot{Q}_5 = \begin{bmatrix} Q_{13} & Q_{14} \\ Q_{14}^T & Q_{15} \end{bmatrix}, \quad \dot{Q}_6 = \begin{bmatrix} Q_{16} & Q_{17} \\ Q_{17}^T & Q_{18} \end{bmatrix},
\]

\[
\dot{Z}_1 = \begin{bmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{bmatrix}, \quad \dot{Z}_2 = \begin{bmatrix} Z_4 & Z_5 \\ Z_5^T & Z_6 \end{bmatrix}, \quad \dot{Z}_3 = \begin{bmatrix} Z_7 & Z_8 \\ Z_8^T & Z_9 \end{bmatrix}, \quad \dot{Z}_4 = \begin{bmatrix} Z_{10} & Z_{11} \\ Z_{11}^T & Z_{12} \end{bmatrix},
\]

\[
\Omega_i = \mathcal{S}_1 + (\varepsilon_1 + \varepsilon_3)\Psi_1^T \Psi_1 + (\varepsilon_2 + \varepsilon_5)\Psi_2^T \Psi_2 + \varepsilon_4 \Psi_3^T \Psi_3 + \varepsilon_6 \Psi_4^T \Psi_4 \quad (i = 1, 2, 3, 4)
\]

where \( \mathcal{S}_1 = (\mathcal{S}_{1k})_{22, 22} \) with

\[
\begin{align*}
\mathcal{S}_{1,1} &= Q_1 + Q_2 - U_1A - A^T U_1^T - \frac{1}{\rho P} (Z_1 + (1 - \mu_1) \mathcal{Z}_1), \\
\mathcal{S}_{1,2} &= Q_2 + Q_8 - \frac{1}{\rho P} (Z_2 + (1 - \mu_2) \mathcal{Z}_2), \\
\mathcal{S}_{1,5} &= \frac{1}{\rho P} (Z_1 + (1 - \mu_2) \mathcal{Z}_1), \\
\mathcal{S}_{1,6} &= \frac{1}{\rho P} (Z_2 + (1 - \mu_2) \mathcal{Z}_2), \\
\mathcal{S}_{1,17} &= P_1 - U_1 - A^T U_1^T, \\
\mathcal{S}_{2,2} &= U_1B, \\
\mathcal{S}_{2,5} &= \frac{1}{\rho P} (Z_2 + (1 - \mu_2) \mathcal{Z}_2), \\
\mathcal{S}_{2,6} &= \frac{1}{\rho P} (Z_3 + (1 - \mu_2) \mathcal{Z}_3), \\
\mathcal{S}_{2,18} &= P_2 - U_5 - A^T U_5^T - M^T X_1^T, \\
\mathcal{S}_{2,22} &= U_5B, \\
\mathcal{S}_{3,3} &= Q_{10} + Q_{16} - U_3 C - C^T U_4^T - \frac{1}{\rho P} (Z_{10} + (1 - \mu_1) \mathcal{Z}_{16}), \\
\mathcal{S}_{3,4} &= Q_{11} + Q_{17} - \frac{1}{\rho P} (Z_{11} + (1 - \mu_1) \mathcal{Z}_{17}), \\
\mathcal{S}_{3,5} &= U_3D, \\
\mathcal{S}_{3,11} &= \frac{1}{\rho P} (Z_{10} + (1 - \mu_1) \mathcal{Z}_{16}), \\
\mathcal{S}_{3,12} &= \frac{1}{\rho P} (Z_{11} + (1 - \mu_1) \mathcal{Z}_{17}), \\
\mathcal{S}_{3,13} &= P_3 - U_3 - C^T U_4^T, \\
\mathcal{S}_{4,4} &= Q_{12} + Q_{18} - U_6 C - C^T U_6^T - X_3 N - N^T X_3^T - \frac{1}{\rho P} (Z_{12} + (1 - \mu_1) \mathcal{Z}_{18}), \\
\mathcal{S}_{4,5} &= U_6D, \\
\mathcal{S}_{4,6} &= \frac{1}{\rho P} (Z_{10} + (1 - \mu_1) \mathcal{Z}_{16}), \\
\mathcal{S}_{4,11} &= \frac{1}{\rho P} (Z_{11} + (1 - \mu_1) \mathcal{Z}_{17}), \\
\mathcal{S}_{4,12} &= \frac{1}{\rho P} (Z_{12} + (1 - \mu_1) \mathcal{Z}_{18}), \\
\mathcal{S}_{4,20} &= P_4 - U_6 - C^T U_6^T - N^T X_3^T, \\
\mathcal{S}_{5,5} &= -(1 - \mu_2) Q_7 - \frac{1}{\rho P} (Z_1 + (1 - \mu_2) \mathcal{Z}_1), \\
\mathcal{S}_{5,6} &= -(1 - \mu_2) Q_8 - \frac{1}{\rho P} (Z_2 + (1 - \mu_2) \mathcal{Z}_2), \\
\mathcal{S}_{5,7} &= \frac{1}{\rho P} Z_1, \\
\mathcal{S}_{5,8} &= \frac{1}{\rho P} Z_2, \\
\mathcal{S}_{5,19} &= D^T U_4^T, \\
\mathcal{S}_{6,6} &= -(1 - \mu_2) Q_9 - \frac{1}{\rho P} (Z_3 + (1 - \mu_2) \mathcal{Z}_3), \\
\mathcal{S}_{6,7} &= \frac{1}{\rho P} Z_3, \\
\mathcal{S}_{6,8} &= \frac{1}{\rho P} Z_2, \\
\mathcal{S}_{7,4} &= -Q_1 + Q_4 - \frac{1}{\rho P} Z_1 - \frac{1}{(1 - \rho P)} Z_4, \\
\mathcal{S}_{7,8} &= -Q_2 + Q_5 - \frac{1}{\rho P} Z_2 - \frac{1}{(1 - \rho P)} Z_5, \\
\mathcal{S}_{7,9} &= \frac{1}{(1 - \rho P)} Z_4, \\
\mathcal{S}_{7,10} &= \frac{1}{(1 - \rho P)} Z_5,
\end{align*}
\]
\[ S_{8,8} = -Q_6 + Q_4 - \frac{1}{(1 - \rho)\pi} Z_3 - \frac{1}{(1 - \rho)\pi} Z_6, \quad S_{8,9} = -Q_5 - \frac{1}{(1 - \rho)\pi} Z_5, \quad S_{8,10} = -Q_6 - \frac{1}{(1 - \rho)\pi} Z_6. \]
\[ S_{9,9} = -Q_4 - \frac{1}{(1 - \rho)\pi} Z_4, \quad S_{9,10} = -Q_5 - \frac{1}{(1 - \rho)\pi} Z_5, \quad S_{10,10} = -Q_6 - \frac{1}{(1 - \rho)\pi} Z_6. \]
\[ S_{11,11} = -(1 - \mu_1) Q_{16} - \frac{1}{\rho \pi} (Z_{10} + (1 - \mu_1) Z_{16}) - \frac{1}{\rho \pi} Z_{10}, \quad S_{11,12} = -(1 - \mu_1) Q_{17} - \frac{1}{\rho \pi} (Z_{11} + (1 - \mu_1) Z_{17}) - \frac{1}{\rho \pi} Z_{11}, \quad S_{11,13} = \frac{1}{\rho \pi} Z_{11}, \quad S_{12,11} = \frac{1}{\rho \pi} Z_{11}, \quad S_{12,12} = \frac{1}{\rho \pi} Z_{12}, \quad S_{12,13} = \frac{1}{\rho \pi} Z_{12}, \quad S_{13,11} = L W_2, \quad S_{13,12} = \frac{1}{\rho \pi} Z_{10} - \frac{1}{(1 - \rho)\pi} Z_{13}. \]
\[ S_{13,14} = -Q_{11} + Q_{14} - \frac{1}{\rho \pi} Z_{11} - \frac{1}{(1 - \rho)\pi} Z_{14}, \quad S_{13,15} = \frac{1}{(1 - \rho)\pi} Z_{13}, \quad S_{13,16} = \frac{1}{(1 - \rho)\pi} Z_{14}. \]
\[ S_{14,14} = -Q_{12} + Q_{15} - \frac{1}{\rho \pi} Z_{12} - \frac{1}{(1 - \rho)\pi} Z_{15}, \quad S_{14,15} = \frac{1}{(1 - \rho)\pi} Z_{14}, \quad S_{14,16} = \frac{1}{(1 - \rho)\pi} Z_{15}. \]
\[ S_{15,15} = -Q_{13} - \frac{1}{(1 - \rho)\pi} Z_{13}, \quad S_{15,16} = -Q_{14} - \frac{1}{(1 - \rho)\pi} Z_{14}, \quad S_{16,16} = -Q_{15} - \frac{1}{(1 - \rho)\pi} Z_{15}. \]
\[ S_{17,17} = \rho Z_{12} + (1 - \rho)Z_{2} + \rho Z_{1} - U_{2} - U_{1}^{T}, \quad S_{17,18} = \rho Z_{2} + (1 - \rho)Z_{3} + \rho Z_{3}, \quad S_{17,19} = \rho Z_{12} + (1 - \rho)Z_{15} + \rho Z_{15} + \rho Z_{16} - U_{6} - U_{6}^{T}. \]
\[ S_{18,18} = \rho Z_{3} + (1 - \rho)Z_{6} + \rho Z_{6} - U_{5} - U_{5}^{T}, \quad S_{18,19} = U_{5} B, \quad S_{19,19} = \rho Z_{10} + (1 - \rho)Z_{13} + \rho Z_{16} - U_{4} - U_{4}^{T}, \quad S_{19,20} = \rho Z_{11} + (1 - \rho)Z_{14} + \rho Z_{17}, \quad S_{20,20} = \rho Z_{12} + (1 - \rho)Z_{15} + \rho Z_{18} + U_{6} - U_{6}^{T}. \]

\[ S_2 = (S_{11,12})^{2,2} \text{ with} \]
\[ S_{11,11}^{1,1} = S_{11,11}, \quad S_{12,12}^{1,1} = S_{12,12}, \quad S_{15,15}^{1,1} = S_{15,15}, \quad S_{16,16}^{1,1} = S_{16,16}, \quad S_{17,17}^{1,1} = S_{17,17}, \quad S_{18,18}^{1,1} = S_{18,18}, \quad S_{19,19}^{1,1} = S_{19,19}. \]
\[ S_{11,12}^{1,1} = S_{11,12}. \]
\[ S_{11,12}^{1,2} = S_{11,12}. \]
\[ S_{11,12}^{1,3} = S_{11,12}. \]
\[ S_{11,12}^{1,4} = S_{11,12}. \]
\[ S_{11,12}^{1,5} = S_{11,12}. \]
\[ S_{11,12}^{1,6} = S_{11,12}. \]
\[ S_{11,12}^{1,7} = S_{11,12}. \]
\[ S_{11,12}^{1,8} = S_{11,12}. \]
\[ S_{11,12}^{1,9} = S_{11,12}. \]
\[ S_{11,12}^{1,10} = S_{11,12}. \]
\[ S_{11,12}^{1,11} = S_{11,12}. \]
\[ S_{11,12}^{1,12} = S_{11,12}. \]
\[ S_{11,12}^{1,13} = S_{11,12}. \]
\[ S_{11,12}^{1,14} = S_{11,12}. \]
\[ S_{11,12}^{1,15} = S_{11,12}. \]
\[ S_{11,12}^{1,16} = S_{11,12}. \]
\[ S_{11,12}^{1,17} = S_{11,12}. \]
\[ S_{11,12}^{1,18} = S_{11,12}. \]
\[ S_{11,12}^{1,19} = S_{11,12}. \]
\[ S_{11,12}^{1,20} = S_{11,12}. \]
\[ S_{11,12}^{1,21} = S_{11,12}. \]
\[ S_{11,12}^{1,22} = S_{11,12}. \]
$$S_{15,15}^{(1)} = S_{15,15}, \quad S_{15,16}^{(1)} = S_{15,16}, \quad S_{16,16}^{(1)} = S_{16,16}, \quad S_{17,17}^{(1)} = S_{17,17}, \quad S_{17,18}^{(1)} = S_{17,18},$$
$$S_{17,21}^{(1)} = S_{17,21}, \quad S_{18,18}^{(1)} = S_{18,18}, \quad S_{18,22}^{(1)} = S_{18,22},$$
$$S_{19,19}^{(1)} = \rho \tau Z_{10} + (1 - \rho) \tau Z_{13} + \tau Z_{16} - U_4 - U_4^f, \quad S_{19,20}^{(1)} = \rho \tau Z_{11} + (1 - \rho) \tau Z_{14} + \tau Z_{17},$$
$$S_{20,20}^{(1)} = \rho \tau Z_{12} + (1 - \rho) \tau Z_{15} + \tau Z_{18} - U_6 - U_6^f, \quad S_{21,21}^{(1)} = S_{21,21}, \quad S_{22,22}^{(1)} = S_{22,22},$$
$$S_3 = (S_{1,2}^{(2)})_{12,22} \text{ with}$$
$$S_{1,1}^{(2)} = S_{1,1}, \quad S_{1,2}^{(2)} = S_{1,2}, \quad S_{1,7}^{(2)} = \frac{1}{\rho \sigma}(Z_1 + (1 - \mu_2)Z_7), \quad S_{1,8}^{(2)} = \frac{1}{\rho \sigma}(Z_2 + (1 - \mu_2)Z_8),$$
$$S_{1,17}^{(2)} = S_{1,17}, \quad S_{2,1}^{(2)} = S_{2,1}, \quad S_{2,2}^{(2)} = S_{2,2}, \quad S_{2,7}^{(2)} = \frac{1}{\rho \sigma}(Z_2 + (1 - \mu_2)Z_8),$$
$$S_{2,8}^{(2)} = \frac{1}{\rho \sigma}(Z_3 + (1 - \mu_2)Z_9), \quad S_{2,18}^{(2)} = S_{2,18}, \quad S_{2,22}^{(2)} = S_{2,22},$$
$$S_{3,3}^{(2)} = S_{3,3}, \quad S_{3,4}^{(2)} = S_{3,4}, \quad S_{3,5}^{(2)} = S_{3,5}, \quad S_{3,11}^{(2)} = S_{3,11}, \quad S_{3,12}^{(2)} = S_{3,12},$$
$$S_{3,19}^{(2)} = S_{3,19}, \quad S_{4,4}^{(2)} = S_{4,4}, \quad S_{4,6}^{(2)} = S_{4,6}, \quad S_{4,11}^{(2)} = S_{4,11}, \quad S_{4,12}^{(2)} = S_{4,12},$$
$$S_{4,20}^{(2)} = S_{4,20}, \quad S_{5,5}^{(2)} = -(1 - \mu_2)Q_7 - \frac{1}{\rho \sigma}(Z_4 + (1 - \mu_2)Z_7) - \frac{1}{(1 - \rho \sigma)Z_4},$$
$$S_{5,5}^{(2)} = -(1 - \mu_2)Q_8 - \frac{1}{\rho \sigma}(Z_5 + (1 - \mu_2)Z_8) - \frac{1}{(1 - \rho \sigma)Z_5},$$
$$S_{5,10}^{(2)} = \frac{1}{\rho \sigma}(Z_4 + (1 - \mu_2)Z_7), \quad S_{5,8}^{(2)} = \frac{1}{\rho \sigma}(Z_4 + (1 - \mu_2)Z_8),$$
$$S_{5,16}^{(2)} = (1 - \mu_2)Z_9, \quad S_{5,19}^{(2)} = S_{5,19},$$
$$S_{5,19}^{(2)} = -(1 - \mu_2)Q_9 - \frac{1}{\rho \sigma}(Z_6 + (1 - \mu_2)Z_9) - \frac{1}{(1 - \rho \sigma)Z_6},$$
$$S_{6,10}^{(2)} = \frac{1}{\rho \sigma}(Z_6 + (1 - \mu_2)Z_9), \quad S_{6,8}^{(2)} = \frac{1}{\rho \sigma}(Z_6 + (1 - \mu_2)Z_9),$$
$$S_{6,9}^{(2)} = \frac{1}{\rho \sigma}(Z_5 + (1 - \mu_2)Z_8), \quad S_{6,10}^{(2)} = \frac{1}{\rho \sigma}(Z_5 + (1 - \mu_2)Z_8),$$
$$S_{6,19}^{(2)} = S_{6,19},$$
$$S_{7,7}^{(2)} = -(Q_1 + Q_4 - \frac{1}{\rho \sigma}(Z_1 + (1 - \mu_2)Z_7) - \frac{1}{(1 - \rho \sigma)Z_4 + (1 - \mu_2)Z_7),$$
$$S_{7,7}^{(2)} = -(Q_2 + Q_4 - \frac{1}{\rho \sigma}(Z_2 + (1 - \mu_2)Z_8) - \frac{1}{(1 - \rho \sigma)Z_4 + (1 - \mu_2)Z_8),$$
$$S_{8,8}^{(2)} = -(Q_3 + Q_6 - \frac{1}{\rho \sigma}(Z_6 + (1 - \mu_2)Z_9), \quad S_{8,8}^{(2)} = -(Q_3 + Q_6 - \frac{1}{\rho \sigma}(Z_6 + (1 - \mu_2)Z_9),$$
$$S_{9,9}^{(2)} = S_{9,9}, \quad S_{9,10}^{(2)} = S_{9,10}, \quad S_{10,10}^{(2)} = S_{10,10}, \quad S_{11,11}^{(2)} = S_{11,11}, \quad S_{11,12}^{(2)} = S_{11,12},$$
$$S_{11,13}^{(2)} = S_{11,13}, \quad S_{11,14}^{(2)} = S_{11,14}, \quad S_{11,21}^{(2)} = S_{11,21}, \quad S_{12,12}^{(2)} = S_{12,12},$$
$$S_{12,13}^{(2)} = S_{12,13}, \quad S_{12,14}^{(2)} = S_{12,14}, \quad S_{12,22}^{(2)} = S_{12,22}, \quad S_{13,13}^{(2)} = S_{13,13},$$
$$S_{13,14}^{(2)} = S_{13,14}, \quad S_{13,15}^{(2)} = S_{13,15}, \quad S_{13,16}^{(2)} = S_{13,16}, \quad S_{14,14}^{(2)} = S_{14,14}, \quad S_{14,15}^{(2)} = S_{14,15},$$
$$S_{14,16}^{(2)} = S_{14,16}, \quad S_{15,15}^{(2)} = S_{15,15}, \quad S_{15,16}^{(2)} = S_{15,16}, \quad S_{16,16}^{(2)} = S_{16,16},$$
$$S_{17,17}^{(2)} = \rho \sigma Z_1 + (1 - \rho) \sigma Z_1 + \sigma Z_4 - U_2 - U_2^f, \quad S_{17,18}^{(2)} = \rho \sigma Z_2 + (1 - \rho) \sigma Z_2 + \sigma Z_4,$$
\[ S_4 = (S_4^{(i)})_{i=1, 2, 3, 4} \]

\[ (S_4^{(i)})_{i=1, 2, 3, 4} \]

\[ \gamma_1 = [U^T_1 \ 0_{1 \times 5}] H_1, \quad \gamma_2 = [0 \ 0 \ U^T_2 \ 0_{1 \times 5} \ U_4 \ 0_{1 \times 3}] H_2, \]

\[ \gamma_3 = [0 \ 0 \ U^T_3 \ 0_{1 \times 5} \ U^T_3 \ 0_{1 \times 15} \ U_4 \ 0_{1 \times 3}] H_3, \quad \gamma_4 = [0 \ X^T_1 \ 0_{1 \times 5} \ X^T_1 \ 0_{1 \times 15} \ X^T_1 \ 0_{1 \times 3}] H_4. \]

References


