Synchronization criteria for coupled stochastic neural networks with time-varying delays and leakage delay

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Abstract

This paper proposes new delay-dependent synchronization criteria for coupled stochastic neural networks with time-varying delays and leakage delay. By constructing a suitable Lyapunov–Krasovskii’s functional and utilizing Finsler’s lemma, novel synchronization criteria for the networks are established in terms of linear matrix inequalities (LMIs) which can be easily solved by using the LMI toolbox in MATLAB. Three numerical examples are given to illustrate the effectiveness of the proposed methods.

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1. Introduction

Complex networks have received increasing attention of researches from various fields of science and engineering such as the World Wide Web, social networks, electrical power grids, global economic markets, and so on [1,2]. Also, in the real applications of systems, there exists naturally time-delay due to the finite information processing speed and the
finite switching speed of amplifiers. It is well known that time-delay often causes undesirable dynamic behaviors such as performance degradation, and instability of the systems. Therefore, recently, the problem of synchronization of coupled neural networks with time-delay which is one of the hot research fields of complex networks has been a challenging issue due to its potential applications such as information science, biological systems and so on [3,4].

On the other hand, in implementation of many practical systems such as aircraft, chemical and biological systems and electric circuits, there exist occasionally stochastic perturbations. It is no less important than the time-delay as a considerable factor affecting dynamics in the systems. Therefore, stochastic modeling with time-delay plays an important role in many fields of science and engineering applications. For this reason, various approaches to stability criteria for stochastic systems with time-delay have been investigated in the literature [5–8]. Xu et al. [5] studied the problem of stability analysis for stochastic systems with parameter uncertainties and a class of nonlinearities. In [6], by the Lyapunov–Krasovskii’s functional based on the delay fractioning approach, an exponential stability criterion for stochastic systems with time-delay was presented. Kwon [7] derived the delay-dependent stability criteria for uncertain stochastic dynamic systems with time-varying delays via the Lyapunov–Krasovskii’s functional approach with two delay fraction numbers. Moreover, by choosing the Lyapunov matrices in the decomposed integral intervals, stability analysis for stochastic neural networks with time-varying delay were proposed in [8]. Above this, the study on the problems for various forms of complex networks or stochastic systems have been addressed. For more details, see the literature [9–16] and references therein.

Recently, the synchronization stability problem for a class of complex dynamical networks with Markovian jumping parameters and mixed time delays had been studied in [17]. The considered model in [17] has stochastic coupling term and stochastic disturbance to reflect more realistic dynamical behaviors of the complex networks that are affected by noisy environment. Very recently, a leakage delay, which is the time delay in leakage term of the systems and a considerable factor affecting dynamics for the worse in the systems, is being put to use in the problem of stability for neural networks [18,19]. Balasubramaniam et al. [18] investigated the problem of passivity analysis for neutral type neural networks with Markovian jumping parameters and time delay in the leakage term. By use of the topological degree theory, delay-dependent stability conditions of neural networks of neutral type with time delay in the leakage term was proposed in [19]. However, to the best of authors’ knowledge, delay-dependent synchronization analysis of coupled stochastic neural networks with time-varying delay and leakage delay has not been investigated yet.

Motivated by the above discussions, the problem of new delay-dependent synchronization criteria for coupled stochastic neural networks with time-varying delays in network coupling and leakage delay is considered. The coupled stochastic neural networks are represented as a simple mathematical model by use of Kronecker product technique. Then, by construction of a suitable Lyapunov–Krasovskii’s functional and utilization of Finsler’s lemma, new synchronization criteria are derived in terms of LMIs which can be solved efficiently by standard convex optimization algorithms [21].
lemma as a tool of getting less conservative synchronization criteria, it should be noted that a new zero equality from the constructed mathematical model is devised. The concept of scaling transformation matrix will be utilized in deriving zero equality of the method. Finally, three numerical examples are included to show the effectiveness of the proposed method.

Notation: $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, and $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrix. For symmetric matrices $X$ and $Y$, $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite (respectively, nonnegative). $I_n$, $0_n$ and $0_{m \times n}$ denote $n \times n$ identity matrix, $n \times n$ and $m \times n$ zero matrices, respectively. $E_n$ denotes the $n \times n$ matrix which all elements are 1. $\| \cdot \|$ refers to the Euclidean vector norm and the induced matrix norm. $\lambda_{\text{max}}(\cdot)$ means the maximum eigenvalue of a given square matrix. $\text{diag}\{\cdot\}$ denotes the block diagonal matrix. $\ast$ represents the elements below the main diagonal of a symmetric matrix. For a given matrix $X \in \mathbb{R}^{n \times n}$, such that $\text{rank}(X) = r$, we define $X^\perp \in \mathbb{R}^{n \times (n-r)}$ as the right orthogonal complement of $X$; i.e., $XX^\perp = 0$. Let $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, P)$ be complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $P$-null sets). $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $P$.

2. Problem statements

Consider the following neural networks with time-varying delays and leakage delay
\[ y(t) = -Ay(t-\tau) + W_1g(y(t)) + W_2g(y(t-h(t))) + b, \]
where $y(t) = [y_1(t), \ldots, y_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector, $n$ denotes the number of neurons in a neural network, $g(\cdot) = [g_1(\cdot), \ldots, g_n(\cdot)]^T \in \mathbb{R}^n$ denotes the neuron activation function vector with $g(0) = 0$, $b = [b_1, \ldots, b_n]^T \in \mathbb{R}^n$ means the constant external input vector, $A = \text{diag}\{a_1, \ldots, a_n\} \in \mathbb{R}^{n \times n}(a_k > 0, k = 1, \ldots, n)$ is the self-feedback matrix, $W_k \in \mathbb{R}^{n \times n}(k = 1, 2)$ are the connection weight matrices, and $h(t)$ and $\tau$ are time-varying delays and leakage delay, respectively, satisfying
\[ 0 \leq h(t) \leq h_M, \hat{h}(t) \leq h_D \quad \text{and} \quad 0 < \tau, \]
where $h_M$ and $\tau$ are positive constant scalars and $h_D$ is any constant one.

In this paper, it is assumed that the activation functions satisfy the following assumption:

Assumption 1. The neurons activation functions, $g_k(\cdot)$, are assumed to be nondecreasing, bounded and globally Lipschitz; that is
\[ \frac{|g_k(\xi_1) - g_k(\xi_2)|}{|\xi_1 - \xi_2|} \leq l_k, \quad \xi_1, \xi_2 \in \mathbb{R}, \quad \xi_1 \neq \xi_2, \quad k = 1, \ldots, n, \]
where $l_k$ are positive constants.

For simplicity, in stability analysis of the network (1), the equilibrium point $y^* = [y_1^*, \ldots, y_n^*]^T$ is shifted to the origin by utilization of the transformation $x(\cdot) = y(\cdot) - y^*$, which leads the system (1) to the following form:
\[ \dot{x}(t) = -Ax(t-\tau) + W_1f(x(t)) + W_2f(x(t-h(t))), \]
where $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the transformed network.
Also, the function $f(\cdot) = [f_1(\cdot), \ldots, f_n(\cdot)]^T$ with $f_q(\cdot) = g_q(\cdot + y^*_q) - g_q(y^*_q)$ and $f_q(0) = 0$ satisfies Assumption 1.

In this paper, a model of coupled stochastic neural networks with time-varying delays in network coupling and leakage delay is considered as

$$
\begin{aligned}
dx_i(t) &= [-Ax_i(t-\tau) + W_1f(x_i(t)) + W_2f(x_i(t-h(t)))] dt + \sum_{j=1}^{N} g_{ij} \Gamma x_j(t-h(t))(dt + d\omega_1(t)) \\
&\quad + \sigma_i(t,x_i(t),x_i(t-h(t)))d\omega_2(t), \quad i = 1, 2, \ldots, N,
\end{aligned}
$$

where $N$ is the number of couple nodes, $x_i(t) = [x_{i1}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n$ is the state vector of the $i$th node, $\Gamma \in \mathbb{R}^{n \times n}$ is the constant inner-coupling matrix of nodes, which represent the individual coupling between the subnetworks, $G = [g_{ij}]_{N \times N}$ is the outer-coupling matrix representing the coupling strength, the topological structure of the networks satisfy the diffusive coupling connections

$$
g_{ij} = g_{ji} \geq 0 \quad (i \neq j), \quad g_{ii} = -\sum_{j=1, i \neq j}^{N} g_{ij} \quad (i, j = 1, 2, \ldots, N),
$$

and $\omega_k(t)(k = 1, 2)$ are $m$-dimensional Wiener processes (Brownian motion) on $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, P)$ which satisfies the following definition.

**Definition 1 (Mörters and Peres [22])**. A real-valued stochastic process $\omega(t)$ is called a Brownian motion or Wiener process if the following holds:

(i) $\omega(0) = 0$ a.s.,
(ii) $\omega(t)$ has independent increment, i.e., for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the increments $\omega(t_2) - \omega(t_1), \ldots, \omega(t_n) - \omega(t_{n-1})$ are independent random variables,
(iii) the increments $\omega(t) - \omega(s)$ are $N(0, t-s)$ for all $0 \leq s \leq t$, where $N(\mu, \sigma^2)$ denote the normal distribution with expected value $\mu$ and variance $\sigma^2$,
(iv) $\omega(t)$ is almost surely continuous.

It should be noted that

$$
\mathbb{E}[d\omega_k(t)] = 0, \quad \mathbb{E}[d\omega_k^2(t)] = dt.
$$

Here, $\omega_1(t)$ and $\omega_2(t)$, which are mutually independent, are the coupling strength disturbance and the system noise, respectively. And the nonlinear uncertainties $\sigma_i(\cdot, \cdot, \cdot) \in \mathbb{R}^{n \times m}(i = 1, \ldots, N)$ are the noise intensity functions satisfying the Lipschitz condition and the following assumption:

$$
\sigma_i^T(t, x_i(t), x_i(t-h(t)))\sigma_i(t, x_i(t), x_i(t-h(t))) \leq \|H_1x_i(t)\|^2 + \|H_2x_i(t-h(t))\|^2,
$$

where $H_k(k = 1, 2)$ are constant matrices with appropriate dimensions.

**Remark 1.** In this paper, the activation functions and the noise intensity functions are assumed to be satisfied the conditions (3) and (6) which are Lipschitz continuous and satisfy the linear growth condition as well. Under the assumptions, according to the work in Chapter 5 of [23], it is easy that the $i$th stochastic neural networks (5) expressed by delayed stochastic differential equation has a unique continuous solution. One can also confirm this fact in [24].
For the convenience of stability analysis for the networks (5), the following Kronecker product and its properties are used.

**Lemma 1** (Graham [25]). Let $\otimes$ denote the notation of Kronecker product. Then, the following properties of the Kronecker product are easily established:

(i) $(zA) \otimes B = A \otimes (zB)$.
(ii) $(A + B) \otimes C = A \otimes C + B \otimes C$.
(iii) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
(iv) $(A \otimes B)^T = A^T \otimes B^T$.

Let us define

\[
x(t) = [x_1(t), \ldots, x_N(t)]^T, \quad f(x(t)) = [f(x_1(t)), \ldots, f(x_N(t))]^T,
\]

\[
\sigma(t) = [\sigma_1(\cdot, \cdot), \ldots, \sigma_N(\cdot, \cdot)]^T.
\]

Then, with Kronecker product in Lemma 1, the network (5) can be represented as

\[
dx(t) = \left[-(I_N \otimes A)x(t-\tau) + (I_N \otimes W_1)f(x(t)) + (I_N \otimes W_2)f(x(t-h(t)))\right] dt
+ (G \otimes \Gamma)x(t-h(t))(dt + d\omega_1(t)) + \sigma(t) d\omega_2(t).
\]

(7)

For stability analysis, Eq. (7) can be rewritten as

\[
dx(t) = \eta(t) dt + \varrho(t) d\omega(t),
\]

(8)

where

\[
\eta(t) = -(I_N \otimes A)x(t-\tau) + (I_N \otimes W_1)f(x(t)) + (I_N \otimes W_2)f(x(t-h(t)))
+ (G \otimes \Gamma)x(t-h(t)),
\]

\[
\varrho(t) = [(G \otimes \Gamma)x(t-h(t)), \sigma(t)], \quad \omega^T(t) = [\omega_1^T(t), \omega_2^T(t)].
\]

The aim of this paper is to investigate the delay-dependent synchronization stability analysis of network (8) with time-varying delays in network coupling and leakage delay. In order to do this, the following definition and lemmas are needed.

**Definition 2** (Liu and Chen [26]). The networks (5) are said to be asymptotically synchronized if the following condition holds:

\[
\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad i, j = 1, 2, \ldots, N.
\]

**Lemma 2** (Cao et al. [3]). Let $U = [u_{ij}]_{N \times N}$, $P \in \mathbb{R}^{n \times n}$, $x^T = [x_1, x_2, \ldots, x_n]^T$, and $y^T = [y_1, y_2, \ldots, y_n]^T$. If $U = U^T$ and each row sum of $U$ is zero, then

\[
x^T(U \otimes P)y = -\sum_{1 \leq i < j \leq N} u_{ij} (x_i - x_j)^T P (y_i - y_j).
\]

**Lemma 3** (Gu [27]). For any constant matrix $M = M^T > 0$, the following inequality holds:

\[
-h(t) \int_{t-h(t)}^t \dot{x}^T(s) M \dot{x}(s) \ ds \leq \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} -M & M \\ -M & -M \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}.
\]

(9)
Lemma 4 (Finsler’s lemma [28]). Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{m \times n}$ such that $\text{rank}(B) < n$. The following statements are equivalent:

(i) $\zeta^T \Phi \zeta < 0, \forall B \zeta = 0, \zeta \neq 0$,

(ii) $B^T \Phi B^T < 0$,

(iii) $\exists X \in \mathbb{R}^{n \times m}$ : $\Phi + XB + B^T X^T < 0$.

3. Main results

In this section, new synchronization criteria for network (8) will be proposed. For the sake of simplicity on matrix representation, $e_i(i = 1, \ldots, 7) \in \mathbb{R}^{7n \times n}$ and $\hat{g}_{ij}$ are defined as block entry matrices (For example, $e_2 = [0, I_n, 0_n, 0_n, 0_n, 0_n, 0_n]^T$) and the $(i,j)$th entry of a matrix $G^T G = [\hat{g}_{ij}]_{N \times N}$, respectively. The notations of several matrices are defined as:

$$
\zeta^T(t) = [x^T(t), x^T(t^-), x^T(t^-h(t)), x^T(t^-h_M), f^T(x(t)), f^T(x(t^-)), \Psi^T(t)],
$$

$$
Y = [0_n, -(I_N \otimes A), (G \otimes \Gamma), 0_n, (I_N \otimes W_1), (I_N \otimes W_2), -I_n],
$$

$$
z_{ij}(t) = x_i(t) - x_j(t), f(z_{ij}(t)) = f(x_i(t)) - f(x_j(t)), \eta_{ij}(t) = \eta_i(t) - \eta_j(t),
$$

$$
\zeta_{ij}^T(t) = [z_{ij}^T(t), z_{ij}^T(t^-), z_{ij}^T(t^-h(t)), z_{ij}^T(t^-h_M), f^T(z_{ij}(t)), f^T(z_{ij}(t^-)), \eta_{ij}^T(t)],
$$

$$
Y_{ij} = [0_n, -A, -(Ng_{ij} \Gamma), 0_n, W_1, W_2, -I_n],
$$

$$
\Xi_1 = e_1P_1e_1^T + e_2P_2e_2^T + e_3(N \hat{g}_{ij} \Gamma^T P_1 \Gamma)e_3^T + e_1(\rho H_1^T H_1)e_1^T + e_3(\rho H_2^T H_2)e_3^T,
$$

$$
\Xi_2 = e_1Q_1 + Q_2e_1^T - e_3(1-h_D)Q_1e_3^T - e_4Q_2e_4^T,
$$

$$
\Xi_3 = e_7(h_M \Gamma R_1)e_7^T - (e_1 - e_3)R(e_1 - e_3)^T - (e_3 - e_4)R(e_3 - e_4)^T - (e_1 - e_3)T(e_3 - e_4)^T - (e_3 - e_4)T(e_1 - e_3)^T,
$$

$$
\Xi_4 = e_1S_1e_1^T - e_2S_1e_2^T + e_7(\tau^2 S_2)e_7^T - e_1S_2e_1^T + e_1S_2e_1^T + e_2S_2e_2^T + e_2S_2e_2^T,
$$

$$
\Xi_5 = e_1L_1^T D_1L_1e_1^T - e_5D_1e_5^T,
$$

$$
\Xi_6 = e_3L_2^T D_2L_2e_3^T - e_6D_2e_6^T,
$$

$$
\Phi = \sum_{i=1}^{6} \Xi_i.
$$

Then, the main results of this paper are presented as follows:

Theorem 1. For given scalars $0 < h_M, h_D$ and $0 < \tau$, the network (8) is asymptotically synchronized for $0 \leq h(t) \leq h_M$ and $h(t) \leq h_D$, if there exist a positive scalar $\rho$, positive definite matrices $P_i, Q_1, Q_2, R, S_1, S_2$, positive diagonal matrices $D_1, D_2$, and any matrix $T$ satisfying the following LMIs for $1 \leq i < j \leq N$:

$$
P_1 - \rho I_n \leq 0,
$$

$$
(11)
$$
where \( Y_{ij} \) and \( \Phi \) are defined in Eq. (10).

**Proof.** Define a matrix \( U \) as

\[
U = [u_{ij}]_{N \times N} = NI_N - E_N.
\]

Consider the following Lyapunov–Krasovskii’s functional candidate as

\[
V = V_1 + V_2 + V_3 + V_4,
\]

where

\[
V_1 = x^T(t)(U \otimes P_1)x(t),
\]

\[
V_2 = \int_{t-h(t)}^{t} x^T(s)(U \otimes Q_1)x(s) \, ds + \int_{t-h_M}^{t} x^T(s)(U \otimes Q_2)x(s) \, ds,
\]

\[
V_3 = h_M \int_{t-h_M}^{t} \int_{s}^{t} \eta^T(u)(U \otimes R)\eta(u) \, du \, ds,
\]

\[
V_4 = \int_{t-\tau}^{t} x^T(s)(U \otimes S_1)x(s) \, ds + \tau \int_{t-\tau}^{t} \int_{s}^{t} \eta^T(u)(U \otimes S_2)\eta(u) \, du \, ds.
\]

By the weak infinitesimal operator \( \mathcal{L} \) in [29] and Lemma 2, the upper bound of \( \mathcal{L}V_1 \) can be obtained as

\[
\mathcal{L}V_1 = 2x^T(t)(U \otimes P_1)\eta(t) + \varphi^T(t)(U \otimes P_1)\varphi(t)
\]

\[
= 2x^T(t)(U \otimes P_1)\eta(t) + \chi^T(t-\overline{h}(t))(G \otimes \Gamma)^T(U \otimes P_1)(G \otimes \Gamma)x(t-\overline{h}(t)) + \sigma^T(t)(U \otimes P_1)\sigma(t)
\]

\[
\leq 2x^T(t)(U \otimes P_1)\eta(t) + \chi^T(t-\overline{h}(t))(G^T NG \otimes \Gamma^T P_1 \Gamma)x(t-\overline{h}(t))
\]

\[
+ \rho \left\{ x^T(t)(U \otimes H_1^T H_1)x(t) + x^T(t-\overline{h}(t))(U \otimes H_2^T H_2)x(t-\overline{h}(t)) \right\}
\]

\[
= \sum_{1 \leq i<j \leq N} \zeta^T_{ij}(t) \Xi_{ij}(t),
\]

where \( \Xi_1 \) is defined in Eq. (10).

Here, by properties of Kronecker product in Lemma 1 and \( UG = GU = NG \), the term \( \Psi_1 \) of \( \mathcal{L}V_1 \) is calculated as

\[
\Psi_1 = x^T(t-\overline{h}(t))(G \otimes \Gamma)^T(UG \otimes P_1 \Gamma)x(t-\overline{h}(t))
\]

\[
= x^T(t-\overline{h}(t))(G^T \otimes \Gamma^T)(NG \otimes P_1 \Gamma)x(t-\overline{h}(t))
\]

\[
= x^T(t-\overline{h}(t))(G^T NG \otimes \Gamma^T P_1 \Gamma)x(t-\overline{h}(t)),
\]

and, if \( P_1 \leq \rho I_n \), then the upper bound of term \( \Psi_2 \) of \( \mathcal{L}V_1 \) is calculated as

\[
\Psi_2 \leq \lambda_{\text{max}}(P_1)(\sigma^T(t)(U \otimes I_n)\sigma(t))
\]

\[
\leq \rho \left( x^T(t)(U \otimes H_1^T H_1)x(t) + x^T(t-\overline{h}(t))(U \otimes H_2^T H_2)x(t-\overline{h}(t)) \right).
\]
Calculating $\mathcal{L}V_2$ gives

$$\mathcal{L}V_2 \leq x^T(t)(U \otimes Q_1)x(t) - (1-h_D)x^T(t-h(t))(U \otimes Q_1)x(t-h(t)) + x^T(t)(U \otimes Q_2)x(t) - x^T(t-h_M)(U \otimes Q_2)x(t-h_M)$$

$$= \sum_{1 \leq i < j \leq N} \zeta^T_{ij}(t) \Xi_{ij}(t),$$

(19)

where $\Xi_2$ is defined in Eq. (10).

From Lemmas 2 and 3 and Theorem 1 in [30], the calculation of $\mathcal{L}V_3$ leads to

$$\mathcal{L}V_3 = h_M^2 \eta^T(t)(U \otimes R) \eta(t)$$

$$- h_M \left( \int_{t-h(t)}^{t} \eta^T(s)(U \otimes R) \eta(s) \, ds + \int_{t-h(t)}^{t} \eta^T(s)(U \otimes R) \eta(s) \, ds \right)$$

$$\leq h_M^2 \eta^T(t)(U \otimes R) \eta(t)$$

$$- \sum_{1 \leq i < j \leq N} \left[ \int_{t-h(t)}^{t} \eta_{ij}(s) \, ds \right]^T \left[ \begin{array}{cc} 1 & 0 \ h(t)h_M^{-1}(U \otimes R) & 0 \ \end{array} \right] \left[ \begin{array}{c} 1 \ h(t)h_M^{-1}(U \otimes R) \end{array} \right] \left[ \begin{array}{c} \int_{t-h(t)}^{t} \eta_{ij}(s) \, ds \end{array} \right]$$

$$\leq h_M^2 \eta^T(t)(U \otimes R) \eta(t)$$

$$- \sum_{1 \leq i < j \leq N} \left[ \int_{t-h(t)}^{t} \eta_{ij}(s) \, ds \right]^T \left[ \begin{array}{cc} R & T \ * & R \end{array} \right] \left[ \begin{array}{c} \int_{t-h(t)}^{t} \eta_{ij}(s) \, ds \ \end{array} \right]$$

$$= h_M^2 \eta^T(t)(U \otimes R) \eta(t)$$

$$- \sum_{1 \leq i < j \leq N} \left[ \int_{t-h(t)}^{t} \eta_{ij}(s) \, ds \right]^T \left[ \begin{array}{cc} U \otimes R & (U \otimes T) \ U \otimes R & (U \otimes T) \end{array} \right] \left[ \begin{array}{c} \int_{t-h(t)}^{t} \eta_{ij}(s) \, ds \ \end{array} \right]$$

(20)

Since the following two zero equalities hold from [31]

$$x(t) - x(t-h(t)) - \int_{t-h(t)}^{t} \eta(s) \, ds - \int_{t-h(t)}^{t} \varphi(s) \, d\omega(s) = 0,$$

(21)

$$x(t-h(t)) - x(t-h_M) - \int_{t-h(t)}^{t-h(t)} \eta(s) \, ds - \int_{t-h(t)}^{t-h(t)} \varphi(s) \, d\omega(s) = 0,$$

(22)

we have

$$-\left( \int_{t-h(t)}^{t} \eta(s) \, ds \right)^T (U \otimes R) \left( \int_{t-h(t)}^{t} \eta(s) \, ds \right)$$

$$\leq - (x(t) - x(t-h(t)))^T (U \otimes R)(x(t) - x(t-h(t)))$$
\[ +2(x(t) - x(t-h(t)))^T (U \otimes R) \int_{t-h(t)}^{t} \varphi(s) \, d\omega(s) \]

\[ - \left( \int_{t-h(t)}^{t} \varphi(s) \, d\omega(s) \right)^T (U \otimes R) \left( \int_{t-h(t)}^{t} \varphi(s) \, d\omega(s) \right), \]  

(23)

\[ - \left( \int_{t-h_M(t)}^{t-h(t)} \eta(s) \, ds \right)^T (U \otimes R) \left( \int_{t-h_M(t)}^{t-h(t)} \eta(s) \, ds \right) \leq -(x(t-h(t)) - x(t-h_M(t)))^T (U \otimes R)(x(t-h(t)) - x(t-h_M(t))) \]

\[ -x(t-h_M(t)) + 2(x(t-h(t)) - x(t-h_M(t)))^T (U \otimes R) \int_{t-h_M(t)}^{t-h(t)} \varphi(s) \, d\omega(s) \]

\[ - \left( \int_{t-h_M(t)}^{t-h(t)} \varphi(s) \, d\omega(s) \right)^T (U \otimes R) \left( \int_{t-h_M(t)}^{t-h(t)} \varphi(s) \, d\omega(s) \right), \]  

(24)

\[ -2 \left( \int_{t-h(t)}^{t} \eta(s) \, ds \right)^T (U \otimes T) \left( \int_{t-h_M(t)}^{t-h(t)} \eta(s) \, ds \right) \]

\[ \leq -2(x(t) - x(t-h(t)))^T (U \otimes T)(x(t-h(t)) - x(t-h_M(t))) \]

\[ -x(t-h_M(t)) + 2(x(t-h(t)) - x(t-h_M(t)))^T (U \otimes T) \int_{t-h_M(t)}^{t-h(t)} \varphi(s) \, d\omega(s) \]

\[ + 2(x(t-h(t)) - x(t-h_M(t)))^T (U \otimes T) \int_{t-h(t)}^{t} \varphi(s) \, d\omega(s) \]

\[ - \left( x(t) - x(t-h(t)) - \int_{t-h(t)}^{t} \eta(s) \, ds \right)^T (U \otimes T) \int_{t-h_M(t)}^{t-h(t)} \varphi(s) \, d\omega(s) \]

\[ - \left( x(t-h(t)) - x(t-h_M(t)) - \int_{t-h_M(t)}^{t-h(t)} \eta(s) \, ds \right)^T (U \otimes T) \int_{t-h_M(t)}^{t-h(t)} \varphi(s) \, d\omega(s). \]  

(25)

It should be noted that

\[ - \left( \int_{t-h(t)}^{t} \varphi(s) \, d\omega(s) \right)^T (U \otimes R) \left( \int_{t-h(t)}^{t} \varphi(s) \, d\omega(s) \right) \leq 0, \]  

(26)

\[ - \left( \int_{t-h_M(t)}^{t-h(t)} \varphi(s) \, d\omega(s) \right)^T (U \otimes R) \left( \int_{t-h_M(t)}^{t-h(t)} \varphi(s) \, d\omega(s) \right) \leq 0. \]  

(27)

Then, an upper bound of \( \mathcal{L}V_3 \) can be rewritten as

\[ \mathcal{L}V_3 \leq \eta^T(t)(U \otimes h_M(t)) \eta(t) - (x(t) - x(t-h(t)))^T (U \otimes R)(x(t-h(t)) - x(t-h_M(t))) \]

\[ -(x(t-h(t)) - x(t-h_M(t)))^T (U \otimes R)(x(t-h(t)) - x(t-h_M(t))) \]

\[ -2(x(t-h(t)) - x(t-h_M(t)))^T (U \otimes T)(x(t-h(t)) - x(t-h_M(t))) + \xi_1(d\omega(t)), \]

\[ = \sum_{1 \leq i < j \leq N} \zeta_{ij}^T(t) \Xi_{ij}(t) + \xi_1(d\omega(t)), \]  

(28)

where \( \Xi_3 \) is defined in Eq. (10), and

\[ \xi_1(d\omega(t)) = 2(x(t) - x(t-h(t)))^T (U \otimes R) \int_{t-h(t)}^{t} \varphi(s) \, d\omega(s) + 2(x(t-h(t)) \]
Calculating $\mathcal{L}V_4$ obtains

$$
\mathcal{L}V_4 \leq x^T(t)(U \otimes S_1)x(t) - x^T(t-\tau)(U \otimes S_1)x(t-\tau)
+ \tau^2 \eta^T(t)(U \otimes S_2)\eta(t) - \left( \int_{t-\tau}^{t} \eta(s) \, ds \right)^T (U \otimes S_2) \left( \int_{t-\tau}^{t} \eta(s) \, ds \right)
\leq x^T(t)(U \otimes S_1)x(t) - x^T(t-\tau)(U \otimes S_1)x(t-\tau)
+ \tau^2 \eta^T(t)(U \otimes S_2)\eta(t) - (x(t)-x(t-\tau))^T (U \otimes S_2)(x(t)-x(t-\tau)) + \xi_2(d\omega(t))
= \sum_{1 \leq i < j \leq N} \xi_{ij}(t) \Xi_4 \xi_{ij}(t) + \xi_2(d\omega(t)),
$$

(29)

where $\Xi_4$ is defined in Eq. (10)

$$
\xi_2(d\omega(t)) = 2(x(t)-x(t-\tau))^T (U \otimes S_2) \int_{t-\tau}^{t} \eta(s) \, ds,
$$

and the similar process of obtaining the upper bound of $\mathcal{L}V_3$ were utilized in calculation of an upper bound of $\mathcal{L}V_4$.

For any diagonal matrices $D_k > 0(k = 1, 2)$ and from Assumption 1, the following inequalities hold:

$$
0 \leq \sum_{1 \leq i < j \leq N} \{ z_{ij}^T(t) L_1^T D_1 L_1 z_{ij}(t) - f^T(z_{ij}(t)) D_1 f(z_{ij}(t)) \}
= \sum_{1 \leq i < j \leq N} \xi_{ij}(t) \Xi_5 \xi_{ij}(t),
$$

(30)

$$
0 \leq \sum_{1 \leq i < j \leq N} \{ z_{ij}^T(t-h(t)) L_2^T D_2 L_2 z_{ij}(t-h(t)) - f^T(z_{ij}(t-h(t))) D_2 f(z_{ij}(t-h(t))) \}
= \sum_{1 \leq i < j \leq N} \xi_{ij}(t) \Xi_6 \xi_{ij}(t),
$$

(31)

where $\Xi_k(k = 5, 6)$ are defined in Eq. (10).

From Eqs. (16)–(31) and by application of S-procedure [21], $\mathcal{L}V$ has a new upper bound as

$$
\mathcal{L}V \leq \sum_{1 \leq i < j \leq N} \xi_{ij}(t) \Phi \xi_{ij}(t) + \xi_1(d\omega(t)) + \xi_2(d\omega(t)),
$$

(32)

where $\Phi$, $\xi_{ij}(t)$ and $\xi_k(d\omega(t))(k = 1, 2)$ are defined in Eqs. (10), (28), and (29).

Also, the network (8) with the augmented vector $\zeta_{ij}(t)$ can be rewritten as

$$
\sum_{1 \leq i < j \leq N} (j-i) Y_{ij} \zeta_{ij}(t) = 0,
$$

(33)

where $Y_{ij}$ is defined as in Eq. (10).
Here, in order to illustrate the process of obtaining Eq. (33), let us define
\[
A = [A_1, A_2, \ldots, A_N] = [N, \ldots, N-1, \ldots, 1] \otimes I_n \in \mathbb{R}^{n \times n N},
\]
where \( A_k(k=1, \ldots, N) \in \mathbb{R}^{n \times n}. \)

By Eqs. (8) and (14) and properties of Kronecker product in Lemma 1, we have the following zero equality:
\[
0 = A(U \otimes I_n) Y(t) = A(U \otimes I_n)[0_{nN}, -(I_N \otimes A), (G \otimes \Gamma), 0_{nN}, (I_N \otimes W_1), (I_N \otimes W_2), -I_{nN}] \zeta(t)
\]
\[
= A[0_{nN}, -(U I_N \otimes I_n A), (U G \otimes I_n \Gamma), 0_{nN}, (U I_N \otimes I_n W_1), (U I_N \otimes I_n W_2), -(U \otimes I_n)] \zeta(t)
\]
\[
= A[0_{nN}, -(U \otimes A), (N G \otimes \Gamma), 0_{nN}, (U \otimes W_1), (U \otimes W_2), -(U \otimes I_n)] \zeta(t)
\]
\[
= -A(U \otimes A)x(t-\tau) + A(N G \otimes \Gamma)x(t-h(t)) + A(U \otimes W_1)f(x(t))
\]
\[+ A(U \otimes W_2)f(x(t-h(t))) - A(U \otimes I_n)\eta(t), \quad (35)
\]
where \( \gamma \) and \( \zeta(t) \) are defined in Eq. (10).

By Lemma 2, the first term of Eq. (35) can be obtained as
\[
-A(U \otimes A)x(t-\tau) = - \left[ \left[ I_n, (N-1) I_n, \ldots, I_n \right] \left( U \otimes A \right) \left[ x_1(t-\tau), \ldots, x_N(t-\tau) \right] \right]^{T}
\]
\[= \sum_{1 \leq i < j \leq N} u_{ij}(A_i - A_j) A(x_i(t-\tau) - x_j(t-\tau))
\]
\[= - \sum_{1 \leq i < j \leq N} (A_i - A_j) A(x_i(t-\tau) - x_j(t-\tau))
\]
\[= - \sum_{1 \leq i < j \leq N} ((N+1-i) I_n) - (N+1-j) I_n\right) A(x_i(t-\tau) - x_j(t-\tau))
\]
\[= - \sum_{1 \leq i < j \leq N} (j-i) I_n A(x_i(t-\tau) - x_j(t-\tau)),
\]
\[= - \sum_{1 \leq i < j \leq N} (j-i) A(x_i(t-\tau) - x_j(t-\tau)). \quad (36)
\]

Similarly, the other terms of Eq. (35) are calculated as
\[
A(N G \otimes \Gamma)x(t-h(t)) = \left[ I_n, \ldots, I_n \right] \left( N G \otimes \Gamma \right) \left[ x_1(t-h(t)), \ldots, x_N(t-h(t)) \right]^{T}
\]
\[= - \sum_{1 \leq i < j \leq N} N g_{ij}(A_i - A_j) \Gamma(x_i(t-h(t)) - x_j(t-h(t)))
\]
\[= - \sum_{1 \leq i < j \leq N} (j-i) (N g_{ij} \Gamma)(x_i(t-h(t)) - x_j(t-h(t))), \quad (37)
\]
\[
A(U \otimes W_1)f(x(t)) = \left[ I_n, \ldots, I_n \right] \left( U \otimes W_1 \right) \left[ f(x_1(t)), \ldots, f(x_N(t)) \right]^{T}
\]
\[= - \sum_{1 \leq i < j \leq N} u_{ij}(A_i - A_j) W_1(f(x_i(t)) - f(x_j(t)))
\]
\[= \sum_{1 \leq i < j \leq N} (j-i) W_1(f(x_i(t)) - f(x_j(t))), \quad (38)
\]
\[
A(U \otimes W_2)f(x(t-h(t))) = \left[ I_n, \ldots, I_n \right] \left( U \otimes W_2 \right) \left[ f(x_1(t-h(t))), \ldots, f(x_N(t-h(t))) \right]^{T}
\]
\[= - \sum_{1 \leq i < j \leq N} u_{ij}(A_i - A_j) W_2(f(x_i(t-h(t))) - f(x_j(t-h(t))))
\]
we have

\[ -\Lambda(U \otimes I_n)\eta(t) = -[N\eta_1, \ldots, I_n]([U \otimes I_n][\eta_1(t), \ldots, \eta_N(t)])^T \]

\[ = \sum_{1 \leq i < j \leq N} u_{ij}(A_i - A_j)I_n(\eta_i(t) - \eta_j(t)) \]

\[ = -\sum_{1 \leq i < j \leq N} (j-i)I_n(\eta_i(t) - \eta_j(t)). \]

Then, Eq. (35) can be rewritten as

\[ 0 = \Lambda(U \otimes I_n)\zeta(t) \]

\[ = -\Lambda(U \otimes A)x(t - \tau) + \Lambda(NG \otimes \Gamma)x(t - h(t)) + \Lambda(U \otimes W_1)f(x(t)) \]

\[ + \Lambda(U \otimes W_2)f(x(t - h(t))) - \Lambda(U \otimes I_n)\eta(t) \]

\[ = -\sum_{1 \leq i < j \leq N} (j-i)A(x_i(t - \tau) - x_j(t - \tau)) \]

\[ - \sum_{1 \leq i < j \leq N} (j-i)(Ng_i \Gamma)(x_i(t-h(t)) - x_j(t-h(t))) \]

\[ + \sum_{1 \leq i < j \leq N} (j-i)W_1(f(x_i(t)) - f(x_j(t))) \]

\[ + \sum_{1 \leq i < j \leq N} (j-i)W_2(f(x_i(t-h(t))) - f(x_j(t-h(t)))) \]

\[ - \sum_{1 \leq i < j \leq N} (j-i)I_n(\eta_i(t) - \eta_j(t)) \]

\[ = \sum_{1 \leq i < j \leq N} (j-i)\gamma_{ij}(t), \] (41)

where \( \gamma_{ij} \) and \( \zeta_{ij}(t) \) are defined in Eq. (10).

Therefore, if Eq. (33) holds, then, a synchronization condition for network (8) is

\[ \sum_{1 \leq i < j \leq N} \gamma_{ij}(t)\Phi_{\zeta_{ij}}(t) < 0 \quad \text{subject to} \quad \sum_{1 \leq i < j \leq N} (j-i)\gamma_{ij}(t) = 0, \] (42)

where \( \Phi \) is defined in Eq. (10).

Here, if the inequality (42) holds, then there exist a positive scalar \( \epsilon \) such that

\[ \Phi < -\epsilon I_{\eta_n}. \] (43)

From Eqs. (32), (42), and (43), by utilization of the mathematical expectation on Eq. (32), we have

\[ \mathbb{E}[LV] \leq \mathbb{E} \left\{ \sum_{1 \leq i < j \leq N} \zeta_{ij}^T(t)\Phi_{\zeta_{ij}}(t) + \xi_1(d\omega(t)) + \xi_2(d\omega(t)) \right\} \]

\[ \leq \mathbb{E} \left\{ \sum_{1 \leq i < j \leq N} \zeta_{ij}^T(t)(-\epsilon I_{\eta_n})\zeta_{ij}(t) \right\} \]

\[ \leq \mathbb{E} \left\{ \sum_{1 \leq i < j \leq N} \zeta_{ij}^T(t)(-\epsilon I_n)\zeta_{ij}(t) \right\}. \]
where $\mathbb{E}(\xi_k(\omega(t))) = 0 (k = 1, 2)$ according to work [5].

By Lyapunov theorem and Definition 1, it can be guaranteed that the subnetworks in the coupled stochastic neural networks (8) are asymptotically synchronized.

Finally, by use of Lemma 4, the condition (42) is equivalent to the following inequality:

$$\sum_{1 \leq i < j \leq N} \{(j-i)Y_{ij}^{-1}\}^T \Phi \{(j-i)Y_{ij}^{-1}\} < 0. \tag{45}$$

From Eq. (45), if the LMIs (13) satisfy, then stability condition (42) holds. This completes our proof. \( \square \)

When the information about $\hat{h}(t)$ is unknown, then by setting $Q_1 = 0_n$ in the Lyapunov–Krasovskii’s functional (15), the following corollary can be obtained.

**Corollary 1.** For given scalars $0 < h_M$ and $0 < \tau$, the network (8) is asymptotically synchronized for $0 \leq h(t) \leq h_M$, if there exist a positive scalar $\rho$, positive definite matrices $P_1$, $Q_2$, $R$, $S_1$, $S_2$, positive diagonal matrices $D_1$, $D_2$, and any matrix $T$ satisfying the following LMIs for $1 \leq i < j \leq N$:

$$P_1 - \rho I_n \leq 0, \tag{46}$$

\[
\begin{bmatrix} R & T \\ * & R \end{bmatrix} \geq 0, \tag{47}
\]

$$\{(j-i)Y_{ij}^{-1}\}^T \Phi \{(j-i)Y_{ij}^{-1}\} < 0, \tag{48}$$

where $Y_{ij}$ is defined in Eq. (10), and $\hat{\Phi} = \Phi - e_1Q_1e_3^T + (1-h_D)e_3Q_1e_3^T$.

**Proof.** The proof of Corollary 1 is very similar to the proof of Theorem 1, so it is omitted. \( \square \)

**Remark 2.** In order to induce a new zero equality (33), the matrix $A$ in Eq. (34) was defined. It is inspired by the concept of scaling transformation matrix. To reduce decision variable, Finsler’s lemma (ii) $B^T \Phi B < 0$ without free-weighting matrices was used. At this time, a zero equality is required. If the matrix $A$ is not considered, then the following description (see only Eq. (36) as example):

$$-\{(U \otimes A)x(t-\tau) = -\{(U \otimes A)[x_1(t-\tau), \ldots, x_N(t-\tau)]^T = \sum_{1 \leq i < j \leq N} \{\} A(x_i(t-\tau) - x_j(t-\tau)) \tag{49}$$

does not hold. Thus, the derivation of zero equality in Eq. (33) is impossible. Here, in order to use Lemma 2, a suitable vector or matrix in the empty parentheses \{\} is needed. Therefore, by defining the matrix $A$, the induction of the zero equality (33) is possible.

**Remark 3.** The equality $A(U \otimes I_N)\zeta(t) = 0$ in Eq. (35) is similar to the concept of eigenvectors for the square matrix. For example, the nonzero vector $x$ is an eigenvector of the square matrix $A$ if there is a scale factor $\lambda$ such that $Ax = \lambda x$. The eigenvector of $A$ are
not changed by the matrix $A$ except for the scale factor $\lambda$. That is, the equality $\dot{\zeta}(t) = A(U \otimes I_n)\zeta(t) = 0$ with $\zeta(t) \neq 0$ can be considered as an operator in the augmented vector $\zeta(t)$. The augmented vector $\zeta(t)$ is also not changed by the operator $Y$ except for the matrix $A(U \otimes I_n)$.

**Remark 4.** In this paper, the problem of new delay-dependent synchronization criteria for coupled stochastic neural networks with leakage delay is considered. By using Finsler’s lemma without free-weighting matrices, the proposed synchronization criteria for the network are formulated in terms of LMIs. Here, as mentioned in Introduction, the leakage delay is the time delay in leakage or forgetting term of the systems and a considerable factor affecting dynamics for the worse in the systems. The effect of the leakage delay which cannot be negligible is shown in Fig. 1. In addition, in the field of delay-dependent stability or synchronization analysis, one of major concerns is to get maximum delay bounds with fewer decision variables [7,32,33]. By utilizing Finsler lemma, one can eliminate free variables which were used in zero equalities in the works [7,33]. From Lemma 4, one can check that the $B_{\perp}^T\Phi B_{\perp} < 0$ is equivalent to the existence of $X$ such that $\Phi + XB + B^TX^T < 0$ holds. Insertion of such an additional matrix $X$ does not play a role to reduce the conservatism of $B_{\perp}^T\Phi B_{\perp} < 0$. It only increases the number of decision variables. Therefore, our proposed synchronization criteria are derived in the form of (ii) in Lemma 4. To do this, the new zero equality (33) with Kronecker product technique was devised, which has not been deduced yet in the literature.

4. Numerical examples

In this section, we provide three numerical examples to illustrate the effectiveness of the proposed synchronization criteria in this paper.

**Example 1.** Consider the following coupled stochastic neural networks with three nodes:

\[
dx_i(t) = [-Ax_i(t - \tau) + W_1 f(x_i(t)) + W_2 f(x_i(t - h(t)))] dt
\]

\[
+ \sum_{j=1}^{3} g_{ij} \Gamma x_j(t - h(t))(dt + d\omega_1(t)) + \sigma_i(t, x_i(t), x_i(t - h(t))) d\omega_2(t),
\]

(50)

where

\[
A = \begin{bmatrix} 3.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -1 & 0.4 \\ 0 & -0.1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.4 & -0.6 \\ 0.2 & 0.4 \end{bmatrix},
\]

\[
\Gamma = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad G = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix},
\]

with

\[
f(x) = \frac{1}{3}(|x + 1| - |x - 1|), \quad L_1 = L_2 = \text{diag}(0.5, 0.5),
\]

\[
\sigma_i^T(t, x_i(t), x_i(t - h(t))) \leq \|x_i(t)\|^2 + \|x_i(t - h(t))\|^2.
\]

For the networks above, the results of maximum bounds of time-delay with different $h_D$ and $\tau$ by Theorem 1 are listed in Table 1. Also, when the value of the time-derivative of...
time-delay is unknown, then by applying Corollary 1 to the above system (50), maximum delay bounds for guaranteeing the asymptotic stability can be obtained as listed in Table 1. In order to confirm the obtained results with the time-varying delay conditions in Table 2, the simulation results for the synchronization errors trajectories, 

\( z_{i1}(t) = x_i(t) - x_1(t) \) 

\( (i = 2, 3) \), of the networks (50) are shown in Figs. 1–3. These figures show that the networks with the errors converge to zero for given initial values of the state by randomly choosing. Specially, the simulation results in Fig. 1 show synchronization errors trajectories for the

Fig. 1. Synchronization errors trajectories with C1-1: (top) \( \tau = 0.05 \), (middle) \( \tau = 0.48 \) and (bottom) \( \tau = 0.49 \) (Example 1).
values of leakage delay, $\tau$, by 0.05, 0.48 and 0.49 with fixed values $h_M = 1.4092$ and $h_D = 0.2$. It is easy to illustrate that the larger value of leakage delay gives the worse dynamic behaviors of the networks.

**Example 2.** Consider the following coupled stochastic neural networks with five nodes:

$$
\begin{align*}
\frac{dx_i(t)}{dt} &= [-Ax_i(t-\tau) + W_1f(x_i(t)) + W_2f(x_i(t-h(t)))]
+ \sum_{j=1}^{5} g_{ij} \Gamma x_j(t-h(t))(dt + d\omega_1(t)) + \sigma_i(t,x_i(t),x_i(t-h(t))) d\omega_2(t),
\end{align*}
$$

(51)
\[ A = \begin{bmatrix} 1.4 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -1.5 & 0.4 \\ 0.1 & -0.2 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.5 & -0.2 \\ 0.1 & -0.5 \end{bmatrix}, \]
\[ G = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \]

where
\[ f(x) = \frac{1}{10}(|x + 1| + |x - 1|), L_1 = L_2 = \text{diag}(0.2, 0.2), \]

\[ \sigma_i^T(t, x_i(t), x_i(t-h(t)))\sigma_i(t, x_i(t), x_i(t-h(t))) \leq \|x_i(t)\|^2 + \|x_i(t-h(t))\|^2. \]

The results of maximum bounds of time-delay with different \( h_D \) and fixed \( \tau = 0.01 \) by Theorem 1 and Corollary 1 are listed in Table 3. Figs. 4–7 show for the state responses, \( x_i(t) (i = 1, 2, 3, 4, 5) \), and the synchronization errors trajectories, \( z_i(t) = x_i(t) - x_1(t) \) \( (i = 2, 3, 4, 5) \) of system (51) with the conditions of time-delay in Table 4 when initial values of the state are \( x_1^T(0) = [1, -1] \), \( x_2^T(0) = [-1.5, 1.2] \), \( x_3^T(0) = [2.0, -1.6] \), \( x_4^T(0) = [0.7, 1.1] \) and \( x_5^T(0) = [-1.2, -1.8] \). Figures show that the system (51) with the errors converge to zero. This implies the synchronization of the network (51).
Fig. 4. State responses with C2-1 (Example 2).

Fig. 5. Synchronization errors trajectories with C2-1 (Example 2).
Fig. 6. State responses with C2-2 (Example 2).

Fig. 7. Synchronization errors trajectories with C2-2 (Example 2).
Example 3. Recall the coupled stochastic neural networks in Example 1 with the following changed parameters:

\[ \Gamma = 2I_n, \quad H_1 = 4I_n, \quad H_2 = H_1. \]

The result of maximum bound of time-delay with \( h_D = 0.2 \) and \( \tau = 0.05 \) by Theorem 1 is 0.0874. With the condition of time-varying delay C3 as \( h(t) = 0.0874\sin^2(2.28t) \) and

Table 4

<table>
<thead>
<tr>
<th>No.</th>
<th>( \tau )</th>
<th>( h_M )</th>
<th>( h_D )</th>
<th>( h(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C2-1</td>
<td>0.01</td>
<td>0.4827</td>
<td>0.2</td>
<td>( 0.2\sin(t) + 0.2827 )</td>
</tr>
<tr>
<td>C2-2</td>
<td>0.01</td>
<td>0.3712</td>
<td>Unknown</td>
<td>( 0.3712</td>
</tr>
</tbody>
</table>

Fig. 8. State responses with C3 (Example 3).

Fig. 9. Synchronization errors trajectories with C3 (Example 3).
\[ \tau = 0.05, \text{ the simulation results for the state responses, } x_i(t) \ (i = 1, 2, 3), \text{ and the synchronization errors trajectories, } z_{ii}(t) = x_i(t) - x_1(t) \ (i = 2, 3), \text{ of the networks are shown in Figs. 8 and 9.} \]

5. Conclusions

In this paper, the delay-dependent synchronization criteria for the coupled stochastic neural networks with time-varying delays in network coupling and leakage delay have been proposed. To do this, a suitable Lyapunov–Krasovskii's functional was used to investigate the feasible region of stability criteria. By establishment of a new zero equality and utilization of Finsler’s lemma, sufficient conditions for guaranteeing asymptotic synchronization for the concerned networks have been derived in terms of LMIs. Three numerical examples have been given to show the effectiveness and usefulness of the presented criteria.

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