

Leader-following Consensus Criteria for Multi-agent Systems with Time-delay and Static Interconnection Topology

Myeongjin Park¹, Ohmin Kwon^{1,*}, Juhyun Park² and Sangmoon Lee³

¹School of Electrical Engineering, Chungbuk National University, Cheongju 361-763, Korea

(Tel : +82-43-261-2422; E-mail: netgauss;madwind@chungbuk.ac.kr)

²Department of Electrical Engineering, Yeungnam University, Kyongsan 712-749, Korea

(Tel : +82-53-810-2491; E-mail: jessie@ynu.ac.kr)

³Department of Electronic Engineering, Daegu University, Gyungsan 712-714, Korea

(Tel : +82-53-850-6647; E-mail: moony@daegu.ac.kr)

Abstract: This paper proposes new delay-dependent consensus criteria for multi-agent systems with time-varying delays and static interconnection topology. By constructing a suitable Lyapunov-Krasovskii's functional and utilizing convex-hull properties, new delay-dependent consensus criteria for the systems are established in terms of linear matrix inequalities (LMIs) which can be easily solved by various effective optimization algorithms. One numerical example is given to illustrate the effectiveness of the proposed methods.

Keywords: Consensus; Multi-agent Systems; Time-delay; Static Interconnection Topology; Lyapunov method; LMI.

1. INTRODUCTION

A multi-agents systems (MASs) have received a great deal of attention due to their extensive applications in many fields such as biology, physics, robotics and control engineering, and so on. A prime concern in this system is the consensus problem. This problem is the agreement of a group of agents on their states of leader by interaction. Specially, consensus problem with the a static leader is called a leader-following consensus problem or consensus regulation. For more details, see the literature [1]-[3] and references therein. Furthermore, it has been paid attention by applications in vehicle systems [4] and networked control systems [5]. During the last few years, the MASs are being put to use in the consensus problem for time-delay which is due to the finite speed of information processing. It is well known that time-delay often causes undesirable dynamic behaviors such as oscillation, performance degradation, and instability of the network. Moreover, analyzing the consensus problem of the MASs with time-delay is identical to investigating the asymptotical stability of ones. The stability criteria for time-delay systems can be classified into two types: delay-dependent ones and delay-independent ones. The former are generally less conservative than the latter because that the delay-dependent stability criterion include the information of delay [6]. Therefore, various approaches to consensus analysis for MASs with time-delay have been investigated in the literature [7]-[11]. By a Lyapunov-based approach and related space decomposition technique, a coordination problem are addressed for the MASs with jointly connected interconnection topologies [7]. Liu *et al.* [8] obtained consensus conditions for the second-order MASs systems with time-

varying communication delays and static interconnection topology. In addition, delay-dependent consensus condition is also obtained for the system with time-invariant communication delay and switching topology. In [9], based on the Lyapunov-Razumikhin methods, the problem of a leader-following consensus for the second-order MASs with identical communications delays was considered. By using the Lyapunov-Krasovskii methods, the consensus conditions are obtained in terms LMIs for the second-order MASs with identical coupling delay [10]. Tian *et al.* [11] studied consensus for the MASs with both communication and input delays. However, less consideration has been paid to the delay-dependent consensus conditions for MASs with time-varying delays.

In this paper, we propose new delay-dependent consensus criteria for MASs with time-varying delays and static interconnection topology. By constructing a suitable Lyapunov-Krasovskii's (L-K) functional and utilizing convex-hull properties, new consensus criteria are derived in terms of LMIs which can be solved efficiently by using the interior-point algorithms [14]. One numerical example is included to show the effectiveness of the proposed method.

Notation: $\mathbf{R}^n$ is the n -dimensional Euclidean space, $\mathbf{R}^{m \times n}$ denotes the set of $m \times n$ real matrix. with the topology of uniform convergence. For symmetric matrices X and Y , $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite (respectively, nonnegative). $I_n \in \mathbf{R}^{n \times n}$ and $0_n \in \mathbf{R}^{n \times n}$ denote the identity matrix and zero matrix, respectively. $0_{m \times n} \in \mathbf{R}^{m \times n}$ denotes the $m \times n$ zero matrix. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. $\text{diag}\{\dots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix. For a given matrix $X \in \mathbf{R}^{m \times n}$, such that $\text{rank}(X) = r$, we define $X^\perp \in \mathbf{R}^{n \times (n-r)}$ as the right orthogonal complement of X ; i.e., $XX^\perp = 0$.

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*Corresponding author.

2. PROBLEM STATEMENTS

The interaction topology of a network of agents is represented using a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the set of nodes $\mathcal{V} = \{1, 2, \dots, n\}$ and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of neighbors of agent i are denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

Consider the following multi-agent systems with second-order agent dynamics

$$\begin{aligned}\dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), \quad i = 1, \dots, n,\end{aligned}\quad (1)$$

where n is the number of nodes, $x_i(t) \in \mathbf{R}$, $v_i(t) \in \mathbf{R}$ and $u_i(t) \in \mathbf{R}$ are the position, velocity and control input, respectively, of agent i .

According to the works [7] and [8], we choose a consensus algorithm with static leader interconnection topology

$$\begin{aligned}u_i(t) &= -k_i v_i(t) - \sum_{j \in \mathcal{N}_i} a_{ij} (x_i(t) - x_j(t)) \\ &\quad - b_i (x_i(t) - x_0),\end{aligned}\quad (2)$$

where $x_0 \in \mathbf{R}$ is the position of the static leader, $k_i > 0$ is control parameter, a_{ij} and b_i are the interconnection weights defining

$$\begin{cases} a_{ij} > 0, & \text{if agent } i \text{ is conneted to agent } j, \\ a_{ij} = 0, & \text{otherwise,} \\ b_i > 0, & \text{if agent } i \text{ is conneted to the leader,} \\ b_i = 0, & \text{otherwise.} \end{cases}$$

In this paper, consider a model of MASs with time-varying delays

$$\begin{aligned}\dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -k_i v_i(t) - \sum_{j \in \mathcal{N}_i} a_{ij} (x_i(t) - x_j(t - h(t))) \\ &\quad - b_i (x_i(t) - x_0),\end{aligned}\quad (3)$$

or

$$\begin{aligned}\dot{x}(t) &= v(t), \\ \dot{v}(t) &= -Kv(t) - (D + B)(x(t) - x_0 1_n) \\ &\quad + A(x(t - h(t)) - x_0 1_n),\end{aligned}\quad (4)$$

where $K = \text{diag}\{k_1, \dots, k_n\}$, $A = [a_{ij}]_{n \times n}$ is a weighted adjacency matrix, $B = \text{diag}\{b_1, \dots, b_n\}$, $D = \text{diag}\{d_1, \dots, d_n\}$ is the degree matrix of $\mathcal{G}$ with $d_i = \sum_{j \neq i} a_{ij}$, $1_n = [1, 1, \dots, 1]^T$ and $h(t)$ is the time-varying delays satisfying

$$0 \leq h(t) \leq h_M, \quad \dot{h}(t) \leq h_D,$$

Let $\hat{x}(t) = x(t) - x_0 1_n$ and $\eta^T(t) = [\hat{x}^T(t), v^T(t)]$, then, the systems (4) can be rewritten as

$$\dot{\eta}(t) = E\eta(t) + F\eta(t - h(t)). \quad (5)$$

where

$$E = \begin{bmatrix} 0_n & I_n \\ -(D + B) & -K \end{bmatrix}, \quad F = \begin{bmatrix} 0_n & 0_n \\ A & 0_n \end{bmatrix}.$$

The aim of this paper is to investigate the delay-dependent consensus analysis (in other word, stability analysis) of the systems (5) with time-varying delays. In order to do this, we introduce the following lemmas.

Lemma 1. For any constant matrix $M = M^T > 0$, the following inequality holds:

$$\begin{aligned}& -h(t) \int_{t-h(t)}^t \dot{x}^T(s) M \dot{x}(s) ds \\ & \leq \begin{bmatrix} x(t) \\ x(t - h(t)) \end{bmatrix}^T \begin{bmatrix} -M & M \\ \star & -M \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - h(t)) \end{bmatrix}. \quad (6)\end{aligned}$$

Proof. According to Jensen's inequality in [13], one obtains

$$\begin{aligned}& (\beta - \alpha) \int_{\beta}^{\alpha} \dot{x}^T(s) M \dot{x}(s) ds \\ & \leq - \left(\int_{\beta}^{\alpha} \dot{x}(s) ds \right)^T M \left(\int_{\beta}^{\alpha} \dot{x}(s) ds \right). \quad (7)\end{aligned}$$

If we choose $\alpha = t$ and $\beta = t - h(t)$ in (7), inequality (6) can be obtained. ■

Lemma 2. [12] Let $\zeta(t) \in \mathbf{R}^n$, $\Phi_{[\cdot]} = \Phi_{[\cdot]}^T \in \mathbf{R}^{n \times n}$, and $\Upsilon \in \mathbf{R}^{m \times n}$ such that $\text{rank}(\Upsilon) < n$. For given $0 \leq h(t) \leq h_M$ where $h(\cdot) : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, the following condition

$$\zeta^T(t) \Phi_{[h(t)]} \zeta(t) < 0, \quad \text{subject to } \Upsilon \zeta(t) = 0$$

holds if the following inequalities:

$$\begin{aligned}\Upsilon^{\perp T} \Phi_{[0]} \Upsilon^{\perp} &< 0, \\ \Upsilon^{\perp T} \Phi_{[h_M]} \Upsilon^{\perp} &< 0.\end{aligned}$$

3. MAIN RESULTS

In this section, we propose new stability criteria for network (5). Before introducing our main results, the notations of several matrices are defined for simplicity:

$$\begin{aligned}\zeta^T(t) &= [\eta^T(t), \eta^T(t - h(t)), \eta^T(t - h_M), \dot{\eta}^T(t)], \\ \Upsilon &= [E, F, 0_{2n}, -I_{2n}], \\ \Phi_{[h(t)]} &= 2e_1 P e_4^T + e_1 (Q_1 + Q_2) e_1^T - e_3 Q_2 e_3^T \\ &\quad - (1 - h_D) e_2 Q_1 e_2^T - 3e_2 R e_2^T + h_M^2 e_4 R e_4^T \\ &\quad - \left(2 - \frac{h(t)}{h_M} \right) e_1 R (e_1^T - e_2^T) \\ &\quad + \left(1 + \frac{h(t)}{h_M} \right) (e_2 - e_3) R e_3^T,\end{aligned}\quad (8)$$

where e_i ($i = 1, \dots, 4$) $\in \mathbf{R}^{8n \times 2n}$ are the corresponding block entry matrices, e.g., $e_2 = [0_{2n}, I_{2n}, 0_{2n}, 0_{2n}]^T$.

Now, we have the following theorem.

Theorem 1. For given scalars $0 < h_M$ and h_D , the agents in the system (5) converge to the state of static leader asymptotically, if there exist positive definite matrices $P \in \mathbf{R}^{2n \times 2n}$, $Q_1 \in \mathbf{R}^{2n \times 2n}$, $Q_2 \in \mathbf{R}^{2n \times 2n}$ and $R \in \mathbf{R}^{2n \times 2n}$ satisfying the following LMIs:

$$\Upsilon^{\perp T} \Phi_{(p)} \Upsilon^{\perp} < 0, \quad p = 1, 2, \quad (9)$$

where $\Phi_{(p)}$ are the vertices of $\Phi_{[h(t)]}$.

Proof. Let us consider the following L-K functional candidate as

$$\begin{aligned} V = & \eta^T(t)P\eta(t) + \int_{t-h(t)}^t \eta^T(s)Q_1\eta(s)ds \\ & + \int_{t-h_M}^t \eta^T(s)Q_2\eta(s)ds \\ & + h_M \int_{t-h_M}^t \int_s^t \dot{\eta}^T(u)R\dot{\eta}(u)duds. \end{aligned} \quad (10)$$

The time-derivative of V can be calculated as

$$\begin{aligned} \dot{V} \leq & 2\eta^T(t)P\dot{\eta}(t) + \eta^T(t)(Q_1 + Q_2)\eta(t) \\ & - (1 - h_D)\eta^T(t - h(t))Q_1\eta(t - h(t)) \\ & - \eta^T(t - h_M)Q_2\eta(t - h_M) + h_M^2\dot{\eta}^T(t)R\dot{\eta}(t) \\ & + \begin{bmatrix} \eta(t) \\ \eta(t - h(t)) \\ \eta(t - h_M) \end{bmatrix}^T \Xi \begin{bmatrix} \eta(t) \\ \eta(t - h(t)) \\ \eta(t - h_M) \end{bmatrix}, \end{aligned} \quad (11)$$

where

$$\Xi = \begin{bmatrix} \left(\frac{h(t)}{h_M} - 2\right)R \left(2 - \frac{h(t)}{h_M}\right)R & 0_{2n} \\ \star & -3R & \left(1 + \frac{h(t)}{h_M}\right)R \\ \star & \star & -\left(1 + \frac{h(t)}{h_M}\right)R \end{bmatrix},$$

and $-h_M = -h(t) - (h_M - h(t))$ and Lemma 1 were utilized in obtaining an upper bound of $\dot{V}$.

From (10) and (11), the time-derivative of V has a new upper bound as

$$\dot{V} \leq \zeta^T(t)\Phi_{[h(t)]}\zeta(t), \quad (12)$$

where $\Phi_{[h(t)]}$ and $\zeta(t)$ are defined in (8).

Also, the system (5) with the augmented vector $\zeta(t)$ can be rewritten as

$$\Upsilon\zeta(t) = 0, \quad (13)$$

where Υ is defined in (8).

Therefore, a stability condition for system (5) is

$$\zeta^T(t)\Phi_{[h(t)]}\zeta(t) < 0, \quad \text{subject to } \Upsilon\zeta(t) = 0. \quad (14)$$

By utilizing Lemma 2, the condition (14) is equivalent to the following LMIs

$$\begin{aligned} \Upsilon^\perp{}^T \Phi_{[0]} \Upsilon^\perp & < 0, \\ \Upsilon^\perp{}^T \Phi_{[h_M]} \Upsilon^\perp & < 0. \end{aligned} \quad (15)$$

From the inequalities (15) and convex-hull properties, if the LMIs (16) satisfy, then stability condition (14) holds. This complete our proof. ■

When the information about $\dot{h}(t)$ is unknown, then, by setting $Q_1 = 0$ in the L-K functional (10), we can obtain the following corollary.

Corollary 1. For a given scalar $0 < h_M$, the agents in the system (5) converge to the state of static leader asymptotically, if there exist positive definite matrices

$P \in \mathbf{R}^{2n \times 2n}$, $Q_2 \in \mathbf{R}^{2n \times 2n}$ and $R \in \mathbf{R}^{2n \times 2n}$ satisfying the following LMIs:

$$\Upsilon^\perp{}^T \hat{\Phi}_{(p)} \Upsilon^\perp < 0, \quad p = 1, 2, \quad (16)$$

where $\hat{\Phi}_{(p)}$ are the vertices of

$$\begin{aligned} \hat{\Phi}_{[h(t)]} = & 2e_1Pe_4^T + e_1Q_2e_1^T - e_3Q_2e_3^T \\ & - 3e_2Re_2^T + h_M^2e_4Re_4^T \\ & - \left(2 - \frac{h(t)}{h_M}\right)e_1R(e_1^T - e_2^T) \\ & + \left(1 + \frac{h(t)}{h_M}\right)(e_2 - e_3)Re_3^T. \end{aligned}$$

4. NUMERICAL EXAMPLES

In this section, one numerical example to illustrate the effectiveness of the proposed stability criteria will be shown.

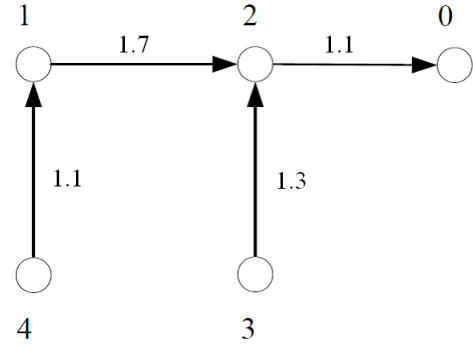


Fig. 1 The static interconnection topology.

Example 1. Consider the MASs (5) with the static interconnection topology described in Figure 1. Clearly, the static leader is the globally reachable. We choose the control gain $K = \text{diag}\{1.5, 1.5, 1.5, 1.5\}$. From Figure 1, the weighted matrix A , B and the degree matrix D are

$$\begin{aligned} A = & \begin{bmatrix} 0 & 1.7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 \\ 1.1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ D = & \text{diag}\{1.7, 0, 1.3, 1.1\}. \end{aligned}$$

For the above networks, the results of the upper bound of time-delay with h_D provided by Theorem 1 and Corollary 1 are listed as Table 1. This means that the proposed criterion can guarantee the network to achieve the asymptotical consensus for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$ any constant $h > 0$ which is less than the upper bound of time-delay with h_D .

It is assumed that the conditions of time-delay are in listed as Table 2. The simulation results for positions and velocities of the agents with the conditions of time-delay are shown in Figures 2 and 3. The figures show that the each agent with the responses converge to the position of static leader for given chosen initial states of the agents.

Table 1 The upper bound of time-delay with h_D .

h_D	0	0.1	0.2	0.5	0.8	unknown
h_M	3.14	2.20	1.87	1.42	1.35	1.35

Table 2 The conditions of Simulation.

No.	Condition	$h(t)$
C1	$h_M = 1.42, h_D = 0.5$	$0.5 \sin(t) + 0.92$
C2	$h_M = 1.35, h_D : \text{unkwoun}$	$1.35 \sin(t) $

5. CONCLUSIONS

In this paper, the delay-dependent consensus criteria for the MASs with time-varying delays and static interconnection topology are proposed. To do this, the suitable L-K functional is used to investigate the feasible region of consensus criteria. One numerical example has been given to show the effectiveness and usefulness of the presented criteria.

REFERENCES

- [1] R.O. Saber, J.A. Fax, R.M. Murray, "Consensus and Cooperation in Networked Multi-Agent Systems," *Proc. The IEEE*, Vol.95, pp.215-233, 2007.
- [2] W. Ren, R.W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Tran. Autom. Control*, Vol.50, pp.655-661, 2005.
- [3] J. Wang, D. Cheng, X. Hu, "Consensus of Multi-agent Linear Dynamic Systems," *Asian J. Control*, Vol.10, pp.144-155, 2008.
- [4] W. Ren, "Consensus strategies for cooperative control of vehicle formations," *IET Control Theory Appl.*, Vol.1, pp.505-512, 2007.
- [5] P. DeLellis, M. DiBernardo, F. Garofalo, D. Liuzza, "Analysis and stability of consensus in networked control systems," *Appl. Math. Comput.*, Vol.217, pp.988-1000, 2010.
- [6] S. Xu, J. Lam, "A survey of linear matrix inequality techniques in stability analysis of delay systems," *Int. J. Syst. Sci.*, Vol.39, pp.1095-1113, 2008.
- [7] Y. Hong, L. Gao, D. Cheng, J. Hu, "Lyapunov-Based Approach to Multiagent Systems With Switching Jointly Connected Interconnection," *IEEE Tran. Autom. Control*, Vol.52, pp.943-948, 2007.
- [8] C.-L. Liu, F. Liu, "Consensus of Second-order Multi-agent Systems under Communication Delay," in: *Proc. Chinses Control Decision Conf.*, pp.739-744, 2010.
- [9] J.P. Hu, Y.G. Hong, "Leader-following coordination of multi-agent systems with coupling time delay," *Physica A*, Vol.374, pp.853-863, 2007.
- [10] H. Su, X. Wang, "Second-order consensus of multiple agents with coupling delay," in: *Proc. the 7th world Congress on Intelligent Control and Automation*, pp.7181-7186, 2008.
- [11] Y.-P. Tian, C.-L. Liu, "Consensus of Mulit-Agnet

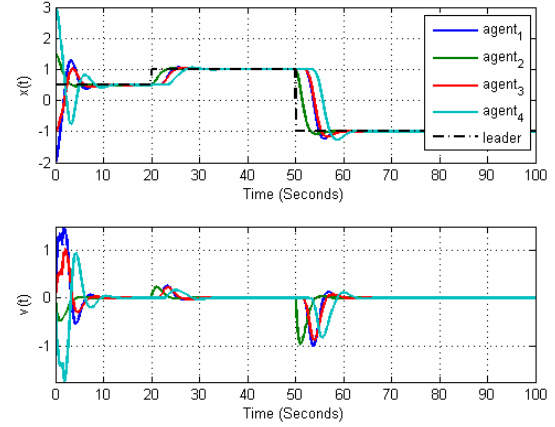


Fig. 2 Positions and velocities of the agents with C1.

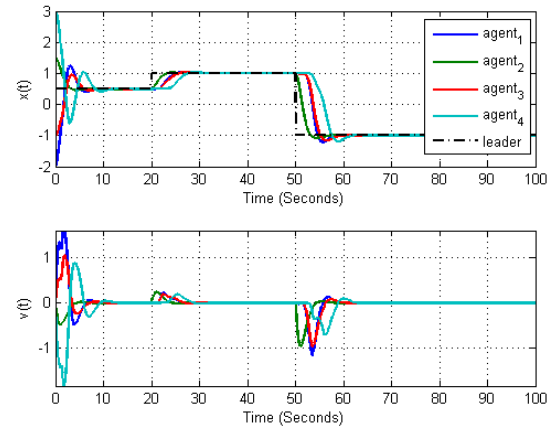


Fig. 3 Positions and velocities of the agents with C2.

- Systems With Diverse Input and Communication Delays," *IEEE Tran. Autom. Control*, Vol.53, pp.2122-2128, 2008.
- [12] M.J. Park, O.M. Kwon, Ju H. Park, S.M. Lee, "A new augmented Lyapunov-Krasovskii functional approach for stability of linear systems with time-varying delays," *Appl. Math. Comput.*, Vol.217, pp.7197-7209, 2011.
- [13] K. Gu, "An integral inequality in the stability problem of time-delay systems," in: *Proc. IEEE Conf. Decision Control*, pp.2805-2810, 2000.
- [14] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.