

Randomly occurring Leader-following Consensus criterion for Multi-agent systems with Communication delay

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Abstract: This paper proposes new delay-dependent leader-following consensus criterion for multi-agent systems with time-varying delays. Specially, the randomly occurring connection information of leader is considered in the concerned system. By constructing a suitable Lyapunov-Krasovskii functional and utilizing reciprocally convex approach, new delay-dependent consensus criterion for the systems are established in terms of linear matrix inequalities (LMIs) which can be easily solved by various effective optimization algorithms. One numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: Randomly occurring leader; Consensus; Multi-agent systems; Time-delay; Lyapunov method.

1. INTRODUCTION

Multi-agents systems (MASs) have received a great deal of attention due to their extensive applications in many fields such as biology, physics, robotics and control engineering, and so on. A prime concern in this system is the consensus problem. This problem is the agreement of a group of agents on their states of leader by interaction. For more details, see the literature [1]-[4] and references therein. Furthermore, it has been paid attention by applications in vehicle systems [2] and networked control systems [3], [4]. Specially, consensus problem with the a leader is called a leader-following consensus problem or consensus regulation [5]-[8]. Song et al. [5] studied the second-order leader-following consensus problem of nonlinear MASs with general network topologies. In particular, this paper addresses what kind of agents and how many agents should be pinned, and establishes some sufficient conditions to guarantee that all agents asymptotically follow the leader. In [6], based on Riccati inequality and Lyapunov inequality, the leader-following consensus problem for MASs with both fixed and switching interaction topologies were considered. Also, the robust analysis of MASs with communication noise was investigated. Chen and Lewis [7] derived distributed controller of a leader-follower network in the presence of communication delays. Moreover, in [8], distributed output regulation of leader-following MASs were purposed. This research is motivated by many practical problems including the target location estimation in a sensor network, and the synchronization of networked agents with the disturbance. However, to the best of author's knowledge, the delay-dependent randomly occurring leader-

following consensus analysis of MASs with time-varying delays has not been investigated yet. Since the finite speed of information processing, it is well known that time-delay often causes undesirable dynamic behaviors such as oscillation and instability of the system. Here, analyzing the consensus problem of the MASs with time-delay is identical to investigating the asymptotical stability of ones. The stability criterion for time-delay systems can be classified into two types: delay-dependent ones and delay-independent ones. The former are generally less conservative than the latter because that the delay-dependent stability criterion include the information of delay [9].

Motivated by the above discussions, we propose new delay-dependent randomly occurring leader-following consensus criterion for MASs with time-varying delays. By constructing a suitable Lyapunov-Krasovskii functional and utilizing reciprocally convex approach [10], new consensus criterion are derived in terms of LMIs which can be formulated as convex optimization algorithms which are amenable to computer solution [11]. One numerical example is included to show the effectiveness of the proposed method.

Notation: $\mathbb{R}^n$ is the n -dimensional Euclidean space, $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrix. For symmetric matrices X and Y , $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite (respectively, nonnegative). $X^\perp$ denotes a basis for the null-space of X . X^T denotes the transposition of X . I denotes the identity matrix with appropriate dimensions. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. $\text{diag}\{\dots\}$ denotes the block diagonal matrix, respectively. $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ denote the expectation of x and the expectation of x conditional on y , respectively.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0009273).
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2. PROBLEM STATEMENTS

The interaction topology of a network of agents is represented using an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with a node set $\mathcal{V} = \{1, \dots, n\}$, an edge set $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\} \subseteq \mathcal{V} \times \mathcal{V}$, and an adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ of a graph is a matrix with nonnegative elements satisfying $a_{ii} = 0$ and $a_{ij} = a_{ji} \geq 0$. If there is an edge between i and j , then the elements of matrix $\mathcal{A}$ described as $a_{ij} = a_{ji} > 0 \Leftrightarrow (i, j) \in \mathcal{E}$. The set of neighbors of node i are denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The degree of node i is denoted by $\deg(i) = \sum_{j \in \mathcal{N}_i} a_{ij}$. The degree matrix of graph $\mathcal{G}$ is diagonal matrix defined as $\mathcal{D} = \text{diag}\{\deg(1), \dots, \deg(n)\}$. The Laplacian matrix $\mathcal{L}$ of graph $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. More details can be seen in [12].

Consider the following multi-agent systems with second-order agent dynamics

$$\begin{aligned} \dot{p}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), \quad i \in \mathcal{V} \end{aligned} \quad (1)$$

where n is the number of nodes, $p_i(t) \in \mathbb{R}^1$, $v_i(t) \in \mathbb{R}^1$ and $u_i(t) \in \mathbb{R}^1$ are the position, velocity and the consensus protocol of agent i , respectively. A consensus algorithm can be described as

$$\begin{aligned} u_i(t) &= -k_i v_i(t) - \sum_{j \in \mathcal{N}_i} a_{ij} (p_i(t) - p_j(t)) \\ &\quad - b_i (p_i(t) - p_0), \end{aligned} \quad (2)$$

where $p_0 \in \mathbb{R}^1$ is the position of the leader, $k_i > 0$ is protocol gain, a_{ij} and b_i are the interconnection weights defining

$$\begin{cases} a_{ij} > 0, & \text{if agent } i \text{ is conneted to agent } j, \\ a_{ij} = 0, & \text{otherwise,} \\ b_i > 0, & \text{if agent } i \text{ is conneted to the leader,} \\ b_i = 0, & \text{otherwise.} \end{cases}$$

With communication delay, a consensus algorithm can be

$$\begin{aligned} u_i(t) &= -k_i v_i(t) - \sum_{j \in \mathcal{N}_i} a_{ij} (p_i(t) - p_j(t - h(t))) \\ &\quad - b_i (p_i(t) - p_0), \end{aligned} \quad (3)$$

where $h(t)$ is the time-varying continuous function satisfying

$$0 \leq h(t) \leq h_M, \quad \dot{h}(t) \leq h_D.$$

Then, the system (1) with consensus algorithm (3) can be rewritten as

$$\begin{aligned} \dot{p}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -k_i v_i(t) - \sum_{j \in \mathcal{N}_i} a_{ij} (p_i(t) - p_j(t - h(t))) \\ &\quad - b_i (p_i(t) - p_0). \end{aligned} \quad (4)$$

It is assumed the interconnection topology is randomly occurring. This mean that, ρ_t is a stochastic process

representing the interconnection topology switching process; that is, let ρ_t be a Bernoulli distributed sequence defined by

$$\rho_t = \begin{cases} 1, & \text{if the information 1 of leader occurs,} \\ 0, & \text{if the information 2 of leader occurs,} \end{cases}$$

where ρ_t satisfies

$$\begin{aligned} \Pr\{\rho_t = 1\} &= \mathbb{E}\{\rho_t\} = \rho_0, \\ \Pr\{\rho_t = 0\} &= 1 - \rho_0. \end{aligned}$$

In this paper, a model of MASSs (4) with randomly occurring interconnection topology are considered as the matrix form

$$\begin{aligned} \dot{p}(t) &= v(t), \\ \dot{v}(t) &= -Kv(t) \\ &\quad - \mathcal{D}(p(t) - p_0 \mathbf{1}_n) + \mathcal{A}(p(t - h(t)) - p_0 \mathbf{1}_n) \\ &\quad - (\rho_t \mathcal{B}^1 + (1 - \rho_t) \mathcal{B}^2)(p(t) - p_0 \mathbf{1}_n), \end{aligned} \quad (5)$$

where

$$\begin{aligned} \mathcal{A} &= [a_{ij}]_{n \times n} \in \mathbb{R}^{n \times n}, \\ \mathcal{B}^m &= \text{diag}\{b_1^m, \dots, b_n^m\} \in \mathbb{R}^{n \times n} \quad (m = 1, 2), \\ \mathcal{D} &= \text{diag}\{d_1, \dots, d_n\} \in \mathbb{R}^{n \times n}, \\ K &= \text{diag}\{k_1, \dots, k_n\} \in \mathbb{R}^{n \times n}, \\ \mathbf{1}_n &= [1, 1, \dots, 1]^T \in \mathbb{R}^{n \times 1}. \end{aligned}$$

For the convenience of consensus analysis for the system (5), let us define

$$\hat{p}(t) = p(t) - p_0 \mathbf{1}_n, \quad x(t) = [\hat{p}^T(t), v^T(t)]^T.$$

Then, the system (4) can be rewritten as

$$\dot{x}(t) = \Lambda(\rho_t)x(t) + \Delta x(t - h(t)), \quad (6)$$

where

$$\begin{aligned} \Lambda(\rho_t) &= \begin{bmatrix} 0 & I \\ -(\mathcal{D} + \rho_t \mathcal{B}^1 + (1 - \rho_t) \mathcal{B}^2) & -K \end{bmatrix}, \\ \Delta &= \begin{bmatrix} 0 & 0 \\ \mathcal{A} & 0 \end{bmatrix}. \end{aligned}$$

The aim of this paper is to investigate the delay-dependent consensus analysis (in other word, stability analysis) of the system (6). In order to do this, we introduce the following lemmas.

Lemma 1. (Jensen's inequality [13]) For any constant matrix $M \in \mathbb{R}^{n \times n}$, $M = M^T > 0$, scalar γ , vector function $x : [0, \gamma] \rightarrow \mathbb{R}^n$ such that the integrations concerned are well defined, then

$$\begin{aligned} & -\gamma \int_0^\gamma x^T(s) M x(s) ds \\ & \leq - \left(\int_0^\gamma x(s) ds \right)^T M \left(\int_0^\gamma x(s) ds \right). \end{aligned}$$

Lemma 2. (Finsler's lemma [14]) Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{m \times n}$ such that $\text{rank}(B) < n$. The following statements are equivalent:

- (i) $\zeta^T \Phi \zeta < 0$, $\forall B\zeta = 0$, $\zeta \neq 0$,
- (ii) $B^{\perp T} \Phi B^{\perp} < 0$.

3. MAIN RESULTS

In this section, we propose new stability criterion for network (6). For simplicity of matrix representation, $e_i \in \mathbb{R}^{8n \times 2n}$ and $e_{i-j} \in \mathbb{R}^{8n \times 2n}$ ($i, j = 1, \dots, 4$) are defined as block entry matrices; e.g., $e_2 = [0, I, 0, 0]^T$ and $e_{1-2} = e_1 - e_2$. The notations of several matrices are defined as:

$$\begin{aligned} \zeta(t) &= [x^T(t), x^T(t-h(t)), x^T(t-h_M), \dot{x}^T(t)]^T, \\ \Upsilon(\rho_t) &= [\Lambda(\rho_t), \Delta, 0, -I], \\ \Phi &= e_1 P e_4^T + e_4 P e_1^T + e_1 Q_1 e_1^T + e_1 Q_2 e_1^T \\ &\quad - (1-h_D) e_2 Q_1 e_2^T - e_3 Q_2 e_3^T \\ &\quad + h_M^2 e_4 R e_4^T - e_{1-2} R e_{1-2}^T - e_{2-3} R e_{2-3}^T \\ &\quad - e_{1-2} S e_{2-3}^T - e_{2-3} S^T e_{1-2}^T. \end{aligned} \quad (7)$$

Now, we have the following theorem.

Theorem 1. For given scalars $0 < h_M, h_D$ and ρ_0 , the agents in the system (6) converge to the state of leader asymptotically, if there exist positive definite matrices $P \in \mathbb{R}^{2n \times 2n}$, $Q_1 \in \mathbb{R}^{2n \times 2n}$, $Q_2 \in \mathbb{R}^{2n \times 2n}$, $R \in \mathbb{R}^{2n \times 2n}$ and any matrix $S \in \mathbb{R}^{2n \times 2n}$ satisfying the following LMIs:

$$[\Upsilon(\rho_0)^\perp]^T \Phi [\Upsilon(\rho_0)^\perp] < 0, \quad (8)$$

$$\begin{bmatrix} R & S \\ S^T & R \end{bmatrix} \geq 0, \quad (9)$$

where Φ and $\Upsilon(\rho_t)$ are defined in (7).

Proof. Let us consider the following Lyapunov-Krasovskii functional candidate as

$$\begin{aligned} V &= x^T(t) P x(t) + \int_{t-h(t)}^t x^T(s) Q_1 x(s) ds \\ &\quad + \int_{t-h_M}^t x^T(s) Q_2 x(s) ds \\ &\quad + h_M \int_{t-h_M}^t \int_s^t \dot{x}^T(u) R \dot{x}(u) du ds. \end{aligned} \quad (10)$$

By the infinitesimal operator $\mathbb{L}$ [15], the $\mathbb{L}V$ can be calculated as

$$\begin{aligned} \mathbb{L}V &\leq 2x^T(t) P \dot{x}(t) + x^T(t) (Q_1 + Q_2) x(t) \\ &\quad - (1-h_D) x^T(t-h(t)) Q_1 x(t-h(t)) \\ &\quad - x^T(t-h_M) Q_2 x(t-h_M) + h_M^2 \dot{x}^T(t) R \dot{x}(t) \\ &\quad - h_M \int_{t-h_M}^t \dot{x}^T(s) R \dot{x}(s) ds. \end{aligned} \quad (11)$$

By Lemma 1, the integral term of $\mathbb{L}V$ bounded as

$$\begin{aligned} &-h_M \int_{t-h_M}^t \dot{x}^T(s) R \dot{x}(s) ds \\ &= -h_M \int_{t-h(t)}^t \dot{x}^T(s) R \dot{x}(s) ds \\ &\quad - h_M \int_{t-h_M}^{t-h(t)} \dot{x}^T(s) R \dot{x}(s) ds \\ &\leq - \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\alpha(t)} R & 0 \\ 0 & \frac{1}{1-\alpha(t)} R \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix}, \end{aligned} \quad (12)$$

where $\alpha(t) = h(t)h_M^{-1}$ and

$$\xi_1(t) = \int_{t-h(t)}^t \dot{x}(s) ds, \quad \xi_2(t) = \int_{t-h_M}^{t-h(t)} \dot{x}(s) ds.$$

Also, by reciprocally convex approach [10], the following inequality for any matrix S and $0 < \alpha(t) < 1$ holds

$$\begin{aligned} &\begin{bmatrix} \sqrt{\frac{1-\alpha(t)}{\alpha(t)}} I & 0 \\ 0 & -\sqrt{\frac{\alpha(t)}{1-\alpha(t)}} I \end{bmatrix} \begin{bmatrix} R & S \\ S^T & R \end{bmatrix} \\ &\times \begin{bmatrix} \sqrt{\frac{1-\alpha(t)}{\alpha(t)}} I & 0 \\ 0 & -\sqrt{\frac{\alpha(t)}{1-\alpha(t)}} I \end{bmatrix} \geq 0, \end{aligned}$$

which implies

$$\begin{bmatrix} \frac{1}{\alpha(t)} R & 0 \\ 0 & \frac{1}{1-\alpha(t)} R \end{bmatrix} \geq \begin{bmatrix} R & S \\ S^T & R \end{bmatrix}.$$

Then, the $\mathbb{E}\{\mathbb{L}V\}$ has a new upper bound as

$$\mathbb{E}\{\mathbb{L}V\} \leq \mathbb{E}\{\zeta^T(t) \Phi \zeta(t)\}, \quad (13)$$

where Φ and $\zeta(t)$ are defined in (7).

Also, the system (6) with the augmented vector $\zeta(t)$ can be rewritten as

$$\mathbb{E}\{\Upsilon(\rho_t) \zeta(t)\} = \Upsilon(\rho_0) \zeta(t) = 0, \quad (14)$$

where $\Upsilon(\rho_t)$ is defined in (7).

Therefore, a stability condition for system (6) is

$$\mathbb{E}\{\zeta^T(t) \Phi \zeta(t)\} < 0 \quad (15)$$

subject to

$$\Upsilon(\rho_0) \zeta(t) = 0.$$

By utilizing Lemma 2, the condition (15) is equivalent to the following LMIs

$$[\Upsilon(\rho_0)^\perp]^T \Phi [\Upsilon(\rho_0)^\perp] < 0. \quad (16)$$

From the inequality (16), if the LMIs (8) satisfy, then stability condition (15) holds. This completes our proof. ■

4. NUMERICAL EXAMPLES

In this section, one numerical example to illustrate the effectiveness of the proposed stability criterion will be shown.

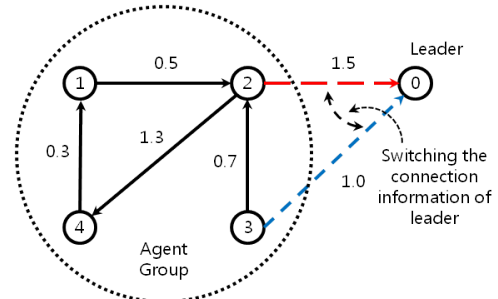


Fig. 1 The interconnection topology.

Example 1. Consider the MASs (6) with the interconnection topology described in Figure 1. Here, the agents

Table 1 The maximum bound of time-delay with different ρ_0 and fixed $h_D = 0.5$.

ρ_0	0	0.3	0.7	1
h_M	infeasible	2.06	2.37	2.61

Table 2 The conditions of Simulation.

No.	ρ_0	h_M	$h(t)$
C1	0.7	2.37	$2.37 \sin^2(0.21t)$

in Figure 1 can be seen in the sense that each agent, e.g., unmanned vehicle and soccer robot, needs information from its local neighborhood. From Figure 1, the related matrices $\mathcal{A}$, $\mathcal{B}^m$ ($m = 1, 2$) and $\mathcal{D}$ are

$$\mathcal{A} = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1.3 \\ 0 & 0.7 & 0 & 0 \\ 0.3 & 0 & 0 & 0 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix},$$

$$\mathcal{B}^1 = \text{diag}\{0, 1.5, 0, 0\},$$

$$\mathcal{B}^2 = \text{diag}\{0, 0, 1.0, 0\}.$$

Also, the protocol gain is assumed as

$$K = \text{diag}\{1, 1, 1, 1\}.$$

For the system above, the results of the maximum bound of time-delay with different ρ_0 and fixed $h_D = 0.5$ provided by Theorem 1 are listed as Table 1. In order to confirm obtained results with the conditions of time-delay in listed as Table 2, the simulation results for positions and velocities of the agents are shown in Figure 2. This figure shows that the each agent with the responses converge to the position of the leader for given chosen initial states of the agents.

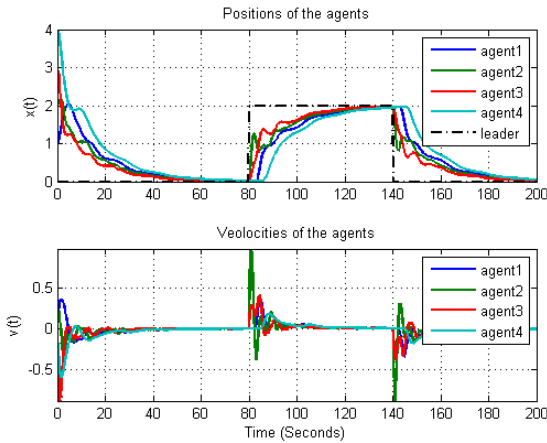


Fig. 2 Positions and velocities of the agents with C1.

5. CONCLUSIONS

In this paper, the delay-dependent randomly occurring leader-following consensus criterion for the MASs with time-varying delays is proposed. To do this, the suitable Lyapunov-Krasovskii functional is used to investigate the

feasible region of consensus criterion. One numerical example has been given to show the effectiveness and usefulness of the presented criterion.

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