Stability for Neural Networks With Time-Varying Delays via Some New Approaches

Oh-Min Kwon, Myeong-Jin Park, Sang-Moon Lee, Ju H. Park, and Eun-Jong Cha

***Abstract*— This paper considers the problem of delay-dependent stability criteria for neural networks with time-varying delays. First, by constructing a newly augmented Lyapunov–Krasovskii functional, a less conservative stability criterion is established in terms of linear matrix inequalities. Second, by proposing novel activation function conditions which have not been proposed so far, further improved stability criteria are proposed. Finally, three numerical examples used in the literature are given to show the improvements over the existing criteria and the effectiveness of the proposed idea.**

***Index Terms*— Lyapunov method, neural networks, stability, time-varying delays.**

I. Introduction

The stability analysis of neural networks is an interesting issue because it can be applied to various fields, including reconstructing a moving image, signal processing, pattern recognition, designing associative memories, fixed-point computations, and other scientific areas [1]–[7]. It is no less important that the equilibrium points of the designed network are stable because the application of neural networks is heavily dependent on the dynamic behavior of the networks. Also, on account of the occurrence of integration and communication delays in the hardware implementation of neural networks, many researchers have devoted time and effort to delay-dependent stability analysis of neural networks with time delays [8]–[27], because it is well known that delay-dependent stability criteria are generally less conservative than delay-independent ones when the size of the time delay is small.

In the field of delay-dependent stability analysis of neural networks, a lot of weight has been placed on the reduction of conservatism of the stability criteria. It is well recognized that an important index for checking the conservatism of stability criteria is to get maximum delay bounds such that the designed networks are asymptotically stable for any delay less than the maximum delay bounds. Therefore, the construction of a suitable Lyapunov–Krasovskii (LK) functional and its estimation calculated by taking the time derivative of the chosen LK functional play key roles in enhancing the feasible region of stability criteria.

To do this, Zhu and Yang [19] proposed a new type of Lyapunov functional to ensure larger delay bounds for neural networks with time-varying delays. By taking more information about time-varying delays and states as augmented vectors and constructing a new LK functional, some new results on stability criteria for neural networks with time-varying delays were proposed [20]. In [21], a novel method, named the delay-slope-dependent method, was proposed by using the fact the neuron activation functions are sector-bounded and nondecreasing.

Recently, to reduce the conservatism of stability criteria in the field of delay-dependent stability analysis, the popular method has been a delay-partitioning method which divides delay interval into some subintervals. As a tradeoff between the time consumed and improvement of the feasible region, the delay-partition number has been chosen as two in many works [22]–[27]. In this regard, in [22] and [23], by utilizing different free-weighting matrices in two delay subintervals, some new methods were proposed to reduce the conservatism of the stability criteria for neural networks with time-varying delays. Recently, by taking a new augmented vector, which includes the information of time-varying delays, a new asymptotic stability criterion was proposed in [24], and its extended result was presented in [25] by constructing a triple integral form of the LK functional to improve the feasible region of stability criteria for neural networks with time-varying delays. Very recently, by utilizing the method of [22], exponential stability of neural networks with interval time-varying delays and general activation functions was investigated in [27]. In [28]–[30], a generalized delay-partitioning method to enhance the feasible region of stability criteria was proposed. One of the main advantages of the methods utilized in [22]–[30] is that they can obtain tighter upper bounds by calculating the time derivative of the LK functional, which leads to less conservative results. However, when the delay-partitioning number increases, the matrix formulation becomes more complex and the computational burden and time consumption grow bigger. It should be noted that, as mentioned in [31], the ability and performance of neural networks are influenced by the choice of the activation functions. Therefore,

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it is natural to look for an alternative view to reduce the conservatism of stability criteria.

Motivated by the above discussion, some new delay-dependent stability criteria for neural networks with time-varying delays in which both the upper and lower bounds of delay derivative are available are proposed in this paper by employing different approaches. The contributions of this paper are threefold.

1) Unlike the method of [22]–[30], no delay-partitioning methods are utilized. Instead, by taking more information of states and activation functions as augmented vectors and constructing a new LK functional, an augmented LK functional is proposed. Then, inspired by the work of [32]–[34], a sufficient condition, such that the considered neural networks are asymptotically stable, is derived in Theorem 1.
2) Based on the result of Theorem 1, a novel approach partitioning the bounding of activation function is proposed in Theorem 2. As a tradeoff between the time consumed and improvement of the feasible region, the bounding of the activation function is divided into two subintervals.
3) With the same LK functional considered in Theorem two, a new activation function condition, which has not been considered so far in the literature, is proposed and utilized in Theorem 3 to reduce the conservatism of the stability criterion.

By utilizing the results of Theorem 3, when only the upper bound of the delay derivative of the time-varying delay is available, the corresponding stability criterion is proposed in Corollary 1. Lastly, when the information about the delay derivative of time-varying delay is unknown, Corollary 2 is presented as a special case of Corollary 1. Through three numerical examples taken from the literature, it is shown that, in spite of not employing delay-partitioning approaches, the proposed stability criteria can provide larger delay bounds than the recent results in which delay-partitioning techniques were utilized.

Notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrix. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices X and Y, the notation $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite, (respectively, nonnegative). diag $\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix. $X_{[f(t)]} \in \mathbb{R}^{m\times n}$ means that the elements of matrix $X_{[f(t)]}$ include the scalar value of $f(t)$.

II. Problem Statements

Consider the following neural networks with discrete time-varying delays:

$$\dot{y}(t) = -Ay(t) + W_0 g(y(t)) + W_1 g(y(t-h(t))) + b \tag{1}$$

where $y(t) = [y_1(t), \ldots, y_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector, n denotes the number of neurons in a neural network, $g(y(t)) = [g_1(y_1(t)), \ldots, g_n(y_n(t))]^T \in \mathbb{R}^n$ means the neuron activation functions, $g(y(t-h(t))) = [g_1(y_1(t-h(t))), \ldots, g_n(y_n(t-h(t)))\ ]^T \in \mathbb{R}^n$, $A = \mathrm{diag}\{a_i\} \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, $W_0 = (w_{ij}^0)_{n\times n} \in \mathbb{R}^{n\times n}$ and $W_1 = (w_{ij}^1)_{n\times n} \in \mathbb{R}^{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons, and $b = [b_1, b_2, \ldots, b_n]^T \in \mathbb{R}^n$ represents a constant input vector.

The delay $h(t)$ is a time-varying continuous function that satisfies the following three cases, where h_U, h_{Dl}, and h_{Du} are known constants.

C1) Time-varying delay: $0 \leq h(t) \leq h_U, h_{Dl} \leq \dot{h}(t) \leq h_{Du} < 1$.
C2) Time-varying delay: $0 \leq h(t) \leq h_U,\ \dot{h}(t) \leq h_{Du}$.
C3) Time-varying delay: $0 \leq h(t) \leq h_U$.

For C1, let us define ∇_d in the following set:

$$\Phi_d := \left\{\nabla_d | \nabla_d \in \mathrm{conv}\left\{\nabla_d^1,\ \nabla_d^2\right\}\right\} \tag{2}$$

where conv denotes the convex hull, $\nabla_d^1 = h_D^l$, and $\nabla_d^2 = h_D^u$. Then, there exists a parameter $\theta > 0$ such that $\dot{h}(t)$ can be expressed as a convex combination of the vertices as follows:

$$\dot{h}(t) = \theta\nabla_d^1 + (1-\theta)\nabla_d^2. \tag{3}$$

If a matrix $M_{[\dot{h}(t)]}$ is affinely dependent on $\dot{h}(t)$, then $M_{[\dot{h}(t)]}$ can be expressed as convex combinations of the vertices

$$M_{[\dot{h}(t)]} = \theta M_{[\nabla_d^1]} + (1-\theta) M_{[\nabla_d^2]}. \tag{4}$$

From (4), if a stability condition is affinely dependent on $\dot{h}(t)$, then it needs only to check at the vertex values of $\dot{h}(t)$ instead of checking all values of $\dot{h}(t)$ [35]. This property will be utilized in Section III.

The neuron activation functions satisfy the following assumption.

Assumption 1: The neuron activation functions $g_i(\cdot)$, $i = 1, \ldots, n$ are continuous, bounded, and satisfy

$$k_i^- \leq \frac{g_i(u) - g_i(v)}{u - v} \leq k_i^+, \qquad u,\ v \in \mathbb{R}$$
$$u \neq v, \qquad i = 1, \ldots, n \tag{5}$$

where k_i^+ and k_i^- are constants.

Remark 1: In Assumption 1, k_i^+ and k_i^- can be allowed to be positive, negative, or zero. As mentioned in [21], Assumption 1 describes the class of globally Lipschitz continuous and monotone nondecreasing activation when $k_i^- = 0$ and $k_i^+ > 0$. And the class of globally Lipschitz continuous and monotone increasing activation functions can be described when $k_i^+ > k_i^- > 0$.

For simplicity, in stability analysis of (1), the equilibrium point $y^* = \left[y_1^*, \ldots, y_n^*\right]^T$ whose uniqueness has been reported in [21] is shifted to the origin by utilizing the transformation $x(\cdot) = y(\cdot) - y^*$, which leads (1) to the following form:

$$\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 f(x(t-h(t))) \tag{6}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the transformed system $f(x(t)) = [f_1(x(t)), \ldots, f_n(x(t))]^T$ and $f_j(x_j(t)) = g_j(x_j(t) + y_j^*) - g_j(y_j^*)$ with $f_j(0) = 0 (j = 1, \ldots, n)$.

It should be noted that the activation functions $f_i(\cdot)$ $(i = 1, \ldots, n)$ satisfy the following condition [15]:

$$k_i^- \leq \frac{f_i(u) - f_i(v)}{u - v} \leq k_i^+, \qquad u,\ v \in \mathbb{R},$$
$$u \neq v, \qquad i = 1, \ldots, n. \tag{7}$$

If $v = 0$ in (7), then we have

$$k_i^- \le \frac{f_i(u)}{u} \le k_i^+ \qquad \forall\, u \ne 0,\ i = 1, \dots, n \tag{8}$$

which is equivalent to

$$\left[f_i(u) - k_i^- u\right]\left[f_i(u) - k_i^+ u\right] \le 0, \qquad i = 1, \dots, n. \tag{9}$$

The objective of this paper is to investigate the delay-dependent stability analysis of (6), which will be done in Section III.

Before deriving our main results, we state the following lemmas.

Lemma 1: For any constant positive-definite matrix $M \in \mathbb{R}^{n\times n}$ and $\beta \le s \le \alpha$, the following inequalities hold:

$$(\alpha-\beta)\int_\beta^\alpha \dot{x}^T(s)M\dot{x}(s)ds \ge \left(\int_\beta^\alpha \dot{x}(s)ds\right)^T M \left(\int_\beta^\alpha \dot{x}(s)ds\right) \tag{10}$$

$$\frac{(\alpha-\beta)^2}{2}\int_\beta^\alpha\int_s^\alpha \dot{x}^T(u)M\dot{x}(u)duds \ge \left(\int_\beta^\alpha\int_s^\alpha \dot{x}(u)duds\right)^T M \left(\int_\beta^\alpha\int_s^\alpha \dot{x}(u)uds\right). \tag{11}$$

Proof: According to Jensen's inequality in [36], one can obtain (10). Moreover, the following inequality holds:

$$(\alpha-s)\int_s^\alpha \dot{x}^T(u)M\dot{x}(u)du \ge \left(\int_s^\alpha \dot{x}(u)du\right)^T M \left(\int_s^\alpha \dot{x}(u)du\right). \tag{12}$$

By Schur complements [37], (12) is equivalent to

$$\begin{bmatrix} \int_s^\alpha \dot{x}^T(u)M\dot{x}(u)du & \int_s^\alpha \dot{x}^T(u)du \\ \int_s^\alpha \dot{x}(u)du & (\alpha-s)M^{-1} \end{bmatrix} \ge 0. \tag{13}$$

Integration of (13) from β to α yields

$$\begin{bmatrix} \int_\beta^\alpha\int_s^\alpha \dot{x}^T(u)M\dot{x}(u)duds & \int_\beta^\alpha\int_s^\alpha \dot{x}^T(u)duds \\ \int_\beta^\alpha\int_s^\alpha \dot{x}(u)duds & \int_\beta^\alpha(\alpha-s)M^{-1}ds \end{bmatrix} \ge 0. \tag{14}$$

Therefore, (14) is equivalent to (11) according to Schur complements. This completes the proof. ■

Lemma 2 [38]: Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$ such that $\text{rank}(B) < n$. Then, the following statements are equivalent:

1) $\zeta^T\Phi\zeta < 0,\ B\zeta = 0,\ \zeta \ne 0$;
2) $(B^\perp)^T\Phi B^\perp < 0$

where $B^\perp$ is a right orthogonal complement of B.

III. Main Results

In this section, new delay-dependent stability criteria for neural networks with time-varying delays (6) are derived. For simplicity of matrix representation, $e_i (i = 1, \dots, 13) \in \mathbb{R}^{13n\times n}$ are defined as block entry matrices. (For example, $e_3^T = [0\ 0\ I\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$). The notations for some matrices are defined as follows:

$$\begin{aligned}
\zeta^T(t) = \Big[&x^T(t)\ \ x^T(t-h(t))\ \ x^T(t-h_U)\ \ \dot{x}^T(t) \\
&\times \dot{x}^T(t-h_U)\ \ \int_{t-h(t)}^t x^T(s)ds \\
&\times \int_{t-h_U}^{t-h(t)} x^T(s)ds\ \ f^T(x(t)) \\
&\times f^T(x(t-h(t)))\ \ f^T(x(t-h_U)) \\
&\times \int_{t-h(t)}^t f^T(x(s))ds\ \ \int_{t-h_U}^{t-h(t)} f^T(x(s))ds \\
&\times \dot{x}^T(t-h(t))\Big]
\end{aligned}$$

$$\begin{aligned}
\Gamma &= \left[-A\ 0\ 0\ -I\ 0\ 0\ 0\ W_0\ W_1 0\ 0\ 0\ 0\right] \\
\alpha^T(t) &= \Big[x^T(t)\ x^T(t-h_U)\ \textstyle\int_{t-h_U}^t x^T(s)ds \\
&\qquad \textstyle\int_{t-h_U}^t f^T(x(s))ds\ x^T(t-h(t))\Big] \\
\beta^T(t) &= \left[x^T(t)\ \dot{x}^T(t)\ f^T(x(t))\right] \\
\Pi_1 &= \left[e_1\ e_3\ e_6+e_7\ e_{11}+e_{12}\ e_2\right] \\
\Pi_2 &= \left[e_4\ e_5\ e_1-e_3\ e_8-e_{10}\ e_{13}\right] \\
\Pi_3 &= \left[e_1\ e_4\ e_8\right],\ \ \Pi_4 = \left[e_3\ e_5\ e_{10}\right] \\
\Pi_5 &= \left[e_2\ e_{13}\ e_9\right] \\
\Pi_6 &= \left[e_6\ e_1-e_2\ e_{11}\ e_7\ e_2-e_3\ e_{12}\right] \\
\Upsilon_{1[\nabla_d^k]} &= \text{diag}\{I, I, I, I, (1-\nabla_d^k)I\} \\
\Phi_1 &= [e_8-e_1K_m]\Lambda_1 e_4^T + e_4\Lambda_1[e_8-e_1K_m]^T \\
&\quad + \left[e_1K_p-e_8\right]\Delta_1 e_4^T + e_4\Delta_1\left[e_1K_p-e_8\right]^T \\
&\quad + [e_{10}-e_3K_m]\Lambda_3 e_5^T + e_5\Lambda_3[e_{10}-e_3K_m]^T \\
&\quad + \left[e_3K_p-e_{10}\right]\Delta_3 e_5^T + e_5\Delta_3\left[e_3K_p-e_{10}\right]^T \\
\Phi_{2[\nabla_d^k]} &= (1-\nabla_d^k)\Big\{[e_9-e_2K_m]\Lambda_2 e_{13}^T \\
&\quad + e_{13}\Lambda_2[e_9-e_2K_m]^T + \left[e_2K_p-e_9\right] \\
&\quad \times \Delta_2 e_{13}^T + e_{13}\Delta_2\left[e_2K_p-e_9\right]^T\Big\} \\
\Xi &= \left(h_U^2/2\right)^2 e_4Q_3e_4^T \\
&\quad -(h_Ue_1-e_6-e_7)Q_3(h_Ue_1-e_6-e_7)^T \\
\Psi &= h_Ue_1Q_4e_1^T + h_Ue_4Q_5e_4^T + e_1P_1e_1^T \\
&\quad + e_2(-P_1+P_2)e_2^T - e_3P_2e_3^T \\
\Theta &= -[e_8-e_1K_m]H_1\left[e_8-e_1K_p\right]^T \\
&\quad - \left[e_8-e_1K_p\right]H_1[e_8-e_1K_m]^T \\
&\quad - [e_9-e_2K_m]H_2\left[e_9-e_2K_p\right]^T \\
&\quad - \left[e_9-e_2K_p\right]H_2[e_9-e_2K_m]^T \\
&\quad - [e_{10}-e_3K_m]H_3\left[e_{10}-e_3K_p\right]^T \\
&\quad - \left[e_{10}-e_3K_p\right]H_3[e_{10}-e_3K_m]^T
\end{aligned}$$

$$\begin{aligned}
\Sigma_{1[\nabla_d^k]} &= \Pi_1\mathcal{R}\Upsilon_{1[\nabla_d^k]}\Pi_2^T + \Pi_2\Upsilon_{1[\nabla_d^k]}^T\mathcal{R}\Pi_1^T + \Pi_3\mathcal{N}\Pi_3^T \\
&\quad -\Pi_4\mathcal{N}\Pi_4^T + \Phi_1 + \Phi_{2[\nabla_d^k]} + \Pi_3\mathcal{Q}_1\Pi_3^T \\
&\quad +(1-\nabla_d^k)\Pi_5(-\mathcal{Q}_1+\mathcal{Q}_2)\Pi_5^T - \Pi_4\mathcal{Q}_2\Pi_4^T \\
&\quad +h_U^2\Pi_3\mathcal{G}\Pi_3^T - \Pi_6\begin{bmatrix}\mathcal{G} & \mathcal{S}\\ \star & \mathcal{G}\end{bmatrix}\Pi_6^T + \Xi + \Psi.
\end{aligned} \tag{15}$$

Now, the following theorem is given by the first main result.

Theorem 1: For a given positive scalar h_U, any one h_{Dl} and h_{Du} with C1, diagonal matrices $K_p = \text{diag}\{k_1^+, \ldots, k_n^+\}$ and $K_m = \text{diag}\{k_1^-, \ldots, k_n^-\}$, (6) is asymptotically stable for $0 \le h(t) \le h_U$ and $h_{Dl} \le \dot{h}(t) \le h_{Du} < 1$ if there exist positive diagonal matrices $\Lambda_i = \text{diag}\{\lambda_{1i}, \ldots, \lambda_{ni}\}$ $(i = 1, 2, 3)$, $\Delta_i = \text{diag}\{\delta_{1i}, \ldots, \delta_{ni}\}$ $(i = 1, 2, 3)$, $H_i = \text{diag}\{h_{1i}, \ldots, h_{ni}\}$ $(i = 1, 2, 3)$, positive definite matrices $\mathcal{R} \in \mathbb{R}^{5n\times 5n}$, $\mathcal{N} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_1 \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_2 \in \mathbb{R}^{3n\times 3n}$, $\mathcal{G} \in \mathbb{R}^{3n\times 3n}$, $Q_i (i = 3, 4, 5) \in \mathbb{R}^{n\times n}$, and any matrix $\mathcal{S} \in \mathbb{R}^{3n\times 3n}$ and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$, satisfying the following linear matrix inequalities (LMIs):

$$\left(\Gamma^{\perp}\right)^T \left(\Sigma_{1[\nabla_d^k]} + \Theta\right)\left(\Gamma^{\perp}\right) < 0 \tag{16}$$

$$\begin{bmatrix} \mathcal{G} & \mathcal{S} \\ \star & \mathcal{G} \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} Q_4 & P_1 \\ \star & Q_5 \end{bmatrix} > 0, \qquad \begin{bmatrix} Q_4 & P_2 \\ \star & Q_5 \end{bmatrix} > 0 \qquad \forall k = 1, 2 \tag{18}$$

where $\Sigma_{1[\nabla_d^k]}$ and Γ are defined in (15), and $\Gamma^{\perp}$ is the right orthogonal complement of Γ.

Proof: For positive diagonal matrices Λ_i, Δ_i $(i = 1, 2, 3)$ and positive definite matrices $\mathcal{R}$, $\mathcal{N}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{G}$, and $Q_i (i = 3, 4, 5)$, let us take the LK functional candidate

$$V = \sum_{i=1}^{7} V_i \tag{19}$$

where

$$\begin{aligned}
V_1 &= \alpha^T(t)\mathcal{R}\alpha(t) \\
V_2 &= \int_{t-h_U}^{t} \beta^T(s)\mathcal{N}\beta(s)ds \\
V_3 &= 2\sum_{i=1}^{n}\left(\lambda_{1i}\int_0^{x_i(t)}(f_i(s) - k_i^- s)ds \right.\\
&\qquad \left. + \delta_{1i}\int_0^{x_i(t)}(k_i^+ s - f_i(s))ds\right) \\
&\quad + 2\sum_{i=1}^{n}\left(\lambda_{2i}\int_0^{x_i(t-h(t))}(f_i(s) - k_i^- s)ds \right.\\
&\qquad \left. + \delta_{2i}\int_0^{x_i(t-h(t))}(k_i^+ s - f_i(s))ds\right) \\
&\quad + 2\sum_{i=1}^{n}\left(\lambda_{3i}\int_0^{x_i(t-h_U)}(f_i(s) - k_i^- s)ds \right.\\
&\qquad \left. + \delta_{3i}\int_0^{x_i(t-h_U)}(k_i^+ s - f_i(s))ds\right) \\
V_4 &= \int_{t-h(t)}^{t} \beta^T(s)\mathcal{Q}_1\beta(s)ds + \int_{t-h_U}^{t-h(t)} \beta^T(s)\mathcal{Q}_2\beta(s)ds \\
V_5 &= h_U \int_{t-h_U}^{t}\int_s^t \beta^T(u)\mathcal{G}\beta(u)duds \\
V_6 &= (h_U^2/2)\int_{t-h_U}^{t}\int_s^t\int_u^t \dot{x}^T(v)Q_3\dot{x}(v)dvduds \\
V_7 &= \int_{t-h_U}^{t}\int_s^t x^T(u)Q_4x(u)duds \\
&\quad + \int_{t-h_U}^{t}\int_s^t \dot{x}^T(u)Q_5\dot{x}(u)duds.
\end{aligned} \tag{20}$$

By the time derivative of V_1, it can be given as

$$\begin{aligned}
\dot{V}_1 &= 2\alpha^T(t)\mathcal{R}\dot{\alpha}(t) \\
&= 2\begin{bmatrix} x(t) \\ x(t-h_U) \\ \int_{t-h(t)}^{t} x(s)ds + \int_{t-h_U}^{t-h(t)} x(s)ds \\ \int_{t-h(t)}^{t} f(x(s))ds + \int_{t-h_U}^{t-h(t)} f(x(s))ds \\ x(t-h(t)) \end{bmatrix}^T \\
&\quad \times\mathcal{R}\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-h_U) \\ x(t) - x(t-h_U) \\ f(x(t)) - f(x(t-h_U)) \\ (1-\dot{h}(t))\dot{x}(t-h(t)) \end{bmatrix} \\
&= \zeta^T(t)\left(\Pi_1\mathcal{R}\Upsilon_{1[\dot{h}(t)]}\Pi_2^T + \Pi_2\Upsilon_{1[\dot{h}(t)]}^T\mathcal{R}\Pi_1^T\right)\zeta(t)
\end{aligned} \tag{21}$$

where

$$\Upsilon_{1[\dot{h}(t)]} = \text{diag}\left\{I, I, I, I, (1-\dot{h}(t))I\right\}. \tag{22}$$

Also, we have

$$\begin{aligned}
\dot{V}_2 &= \beta^T(t)\mathcal{N}\beta(t) - \beta^T(t-h_U)^T\mathcal{N}\beta^T(t-h_U) \\
&= \zeta^T(t)\left[\Pi_3\mathcal{N}\Pi_3^T - \Pi_4\mathcal{N}\Pi_4^T\right]\zeta(t).
\end{aligned} \tag{23}$$

Calculation of $\dot{V}_3$ gives

$$\begin{aligned}
\dot{V}_3 &= 2\left[f(x(t)) - K_m x(t)\right]^T \Lambda_1\dot{x}(t) \\
&\quad + 2\left[K_p x(t) - f(x(t))\right]^T \Delta_1\dot{x}(t) \\
&\quad + (1-\dot{h}(t))\left\{2\left[f(x(t-h(t))) - K_m x(t-h(t))\right]^T\right. \\
&\qquad\qquad \times\Lambda_2\dot{x}(t-h(t)) \\
&\qquad\qquad + 2\left[K_p x(t-h(t)) - f(x(t-h(t)))\right]^T \\
&\qquad\qquad \left.\times\Delta_2\dot{x}(t-h(t))\right\} \\
&\quad + 2\left[f(x(t-h_U)) - K_m x(t-h_U)\right]^T \Lambda_3\dot{x}(t-h_U) \\
&\quad + 2\left[K_p x(t-h_U) - f(x(t-h_U))\right]^T \Delta_3\dot{x}(t-h_U) \\
&= \zeta^T(t)\left(\Phi_1 + \Phi_{2[\dot{h}(t)]}\right)\zeta(t)
\end{aligned} \tag{24}$$

where Φ_1 was defined in (15) and

$$\begin{aligned}
\Phi_{2[\dot{h}(t)]} = (1-\dot{h}(t))&\left\{\left[e_9 - e_2K_m\right]\Lambda_2 e_{13}^T\right. \\
&\quad + e_{13}\Lambda_2\left[e_9 - e_2K_m\right]^T \\
&\quad + \left[e_2K_p - e_9\right]\Delta_2 e_{13}^T \\
&\quad \left. + e_{13}\Delta_2\left[e_2K_p - e_9\right]^T\right\}.
\end{aligned} \tag{25}$$

Calculation of $\dot{V}_4$ leads to

$$\begin{aligned}\dot{V}_4 &= \beta^T(t)\mathcal{Q}_1\beta(t)\\ &\quad -(1-\dot{h}(t))\beta^T(t-h(t))\mathcal{Q}_1\beta(t-h(t))\\ &\quad +(1-\dot{h}(t))\beta^T(t-h(t))\mathcal{Q}_2\beta(t-h(t))\\ &\quad -\beta^T(t-h_U)\mathcal{Q}_2\beta(t-h_U)\\ &= \zeta^T(t)\Big[\Pi_3\mathcal{Q}_1\Pi_3^T + (1-\dot{h}(t))\Pi_5(-\mathcal{Q}_1+\mathcal{Q}_2)\Pi_5^T\\ &\qquad -\Pi_4\mathcal{Q}_2\Pi_4^T\Big]\zeta(t). \qquad (26)\end{aligned}$$

By use of Lemma 1 introduced in Section II and Theorem 1 in [33], if (17) holds, then an estimation of $\dot{V}_5$ can be obtained as

$$\begin{aligned}\dot{V}_5 &= h_U^2\beta^T(t)\mathcal{G}\beta(t) - h_U\int_{t-h(t)}^{t}\beta^T(s)\mathcal{G}\beta(s)ds\\ &\quad -h_U\int_{t-h_U}^{t-h(t)}\beta^T(s)\mathcal{G}\beta(s)ds\\ &\le h_U^2\beta^T(t)\mathcal{G}\beta(t) - \left(\frac{h_U}{h(t)}\right)\left(\int_{t-h(t)}^{t}\beta(s)ds\right)^T\mathcal{G}\\ &\quad \times\left(\int_{t-h(t)}^{t}\beta(s)ds\right) - \left(\frac{h_U}{h_U-h(t)}\right)\\ &\quad \times\left(\int_{t-h_U}^{t-h(t)}\beta(s)ds\right)^T\mathcal{G}\times\left(\int_{t-h_U}^{t-h(t)}\beta(s)ds\right)\\ &\le h_U^2\beta^T(t)\mathcal{G}\beta(t) - \begin{bmatrix}\int_{t-h(t)}^{t}\beta(s)ds\\ \int_{t-h_U}^{t-h(t)}\beta(s)ds\end{bmatrix}^T\begin{bmatrix}\mathcal{G} & \mathcal{S}\\ \star & \mathcal{G}\end{bmatrix}\\ &\quad \times\begin{bmatrix}\int_{t-h(t)}^{t}\beta(s)ds\\ \int_{t-h_U}^{t-h(t)}\beta(s)ds\end{bmatrix}\\ &= \zeta^T(t)\left\{h_U^2\Pi_3\mathcal{G}\Pi_3^T - \Pi_6\begin{bmatrix}\mathcal{G} & \mathcal{S}\\ \star & \mathcal{G}\end{bmatrix}\Pi_6^T\right\}\zeta(t). \qquad (27)\end{aligned}$$

For the detailed proof of (27), see [39].

By Lemma 1, $\dot{V}_6$ is bounded as

$$\begin{aligned}\dot{V}_6 &= (h_U^2/2)^2\dot{x}^T(t)Q_3\dot{x}(t)\\ &\quad -(h_U^2/2)\int_{t-h_U}^{t}\int_{s}^{t}\dot{x}^T(u)Q_3\dot{x}(u)duds\\ &\le (h_U^2/2)^2\dot{x}^T(t)Q_3\dot{x}(t) - \left(\int_{t-h_U}^{t}\int_{s}^{t}\dot{x}(u)duds\right)^T\\ &\quad \times Q_3\left(\int_{t-h_U}^{t}\int_{s}^{t}\dot{x}(u)duds\right)\\ &= (h_U^2/2)^2\dot{x}^T(t)Q_3\dot{x}(t)\\ &\quad -\left(h_Ux(t)-\int_{t-h_U}^{t}x(s)ds\right)^T\\ &\quad \times Q_3\left(h_Ux(t)-\int_{t-h_U}^{t}x(s)ds\right)\\ &= (h_U^2/2)^2\dot{x}^T(t)Q_3\dot{x}(t)\\ &\quad -\left(h_Ux(t)-\int_{t-h(t)}^{t}x(s)ds-\int_{t-h_U}^{t-h(t)}x(s)ds\right)^T\\ &\quad \times Q_3\left(h_Ux(t)-\int_{t-h(t)}^{t}x(s)ds-\int_{t-h_U}^{t-h(t)}x(s)ds\right)\\ &= \zeta^T(t)\Xi\zeta(t). \qquad (28)\end{aligned}$$

Finally, $\dot{V}_7$ is easily obtained as

$$\begin{aligned}\dot{V}_7 &= h_Ux^T(t)Q_4x(t) - \int_{t-h_U}^{t}x^T(s)Q_4x(s)ds\\ &\quad +h_U\dot{x}^T(t)Q_5\dot{x}(t) - \int_{t-h_U}^{t}\dot{x}^T(s)Q_5\dot{x}(s)ds. \qquad (29)\end{aligned}$$

Inspired by the work of [34], the following two zero equalities with any symmetric matrices P_1 and P_2 are considered:

$$\begin{aligned}0 &= x^T(t)P_1x(t) - x^T(t-h(t))P_1x(t-h(t))\\ &\quad -2\int_{t-h(t)}^{t}x^T(s)P_1\dot{x}(s)ds\\ 0 &= x^T(t-h(t))P_2x(t-h(t)) - x^T(t-h_U)P_2x(t-h_U)\\ &\quad -2\int_{t-h_U}^{t-h(t)}x^T(s)P_2\dot{x}(s)ds. \qquad (30)\end{aligned}$$

With the zero equalities, an upper bound of $\dot{V}_7$ is

$$\begin{aligned}\dot{V}_7 &\le \zeta^T(t)\Psi\zeta(t)\\ &\quad -\int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix}Q_4 & P_1\\ \star & Q_5\end{bmatrix}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds\\ &\quad -\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T\begin{bmatrix}Q_4 & P_2\\ \star & Q_5\end{bmatrix}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds. \qquad (31)\end{aligned}$$

If (18) hold, then

$$\dot{V}_7 \le \zeta^T(t)\Psi\zeta(t). \qquad (32)$$

From (8), for any positive diagonal matrices $H_i = \mathrm{diag}\{h_{1i},\ldots,h_{ni}\}$ $(i = 1,2,3)$, the following inequality holds:

$$\begin{aligned}0 &\le -2\sum_{i=1}^{n}h_{i1}\left[f_i(x_i(t)) - k_i^-x_i(t)\right]\\ &\quad \times\left[f_i(x_i(t)) - k_i^+x_i(t)\right]\\ &\quad -2\sum_{i=1}^{n}h_{i2}\left[f_i(x_i(t-h(t))) - k_i^-x_i(t-h(t))\right]\\ &\quad \times\left[f_i(x_i(t-h(t))) - k_i^+x_i(t-h(t))\right]\\ &\quad -2\sum_{i=1}^{n}h_{i3}\left[f_i(x_i(t-h_U)) - k_i^-x_i(t-h_U)\right]\\ &\quad \times\left[f_i(x_i(t-h_U)) - k_i^+x_i(t-h_U)\right]\\ &= \zeta^T(t)\Theta\zeta(t). \qquad (33)\end{aligned}$$

From (19)–(33) and by application of the S-procedure [37], if (18) holds, then an upper bound of $\dot{V}$ is

$$\dot{V} \le \zeta^T(t)\left(\Sigma_{1[\dot{h}(t)]} + \Theta\right)\zeta(t) \qquad (34)$$

where

$$\begin{aligned}\Sigma_{1[\dot{h}(t)]} &= \Pi_1\mathcal{R}\Upsilon_{1[\dot{h}(t)]}\Pi_2^T + \Pi_2\Upsilon_{1[\dot{h}(t)]}^T\mathcal{R}\Pi_1^T + \Pi_3\mathcal{N}\Pi_3^T\\ &\quad -\Pi_4\mathcal{N}\Pi_4^T + \Phi_1 + \Phi_{2[\dot{h}(t)]} + \Pi_3\mathcal{Q}_1\Pi_3^T\\ &\quad +(1-\dot{h}(t))\Pi_5(-\mathcal{Q}_1+\mathcal{Q}_2)\Pi_5^T - \Pi_4\mathcal{G}_2\Pi_4^T\\ &\quad +h_U^2\Pi_3\mathcal{G}\Pi_3^T - \Pi_6\begin{bmatrix}\mathcal{G} & \mathcal{S}\\ \star & \mathcal{G}\end{bmatrix}\Pi_6^T + \Xi + \Psi. \qquad (35)\end{aligned}$$

It should be noted that $\Sigma_{1[\dot{h}(t)]}$ is affinely dependent on $\dot{h}(t)$. By Lemma 2, $\zeta^T(t)(\Sigma_{1[\dot{h}(t)]}+\Theta)\zeta(t) < 0$ with $0 = \Gamma\zeta(t)$ is equivalent to $(\Gamma^{\perp})^T(\Sigma_{1[\dot{h}(t)]}+\Theta)\Gamma^{\perp} < 0$. Thus, if (16) for $k = 1, 2$, (17), and (18) hold, then (6) is asymptotically stable for $0 \le h(t) \le h_U$ and $h_{Dl} \le \dot{h}(t) \le h_{Du}$. This completes our proof. ■

Remark 2: Recently, the reciprocally convex optimization technique to reduce the conservatism of stability criteria for systems with time-varying delays was proposed in [33]. Motivated by this paper, the proposed method of [33] was utilized in (27). In Theorem 1, an augmented vector $\zeta(t)$ including the integral terms $\int_{t-h(t)}^{t} x(s)ds$, $\int_{t-h_U}^{t-h(t)} x(s)ds$, $\int_{t-h(t)}^{t} f(x(s))ds$, $\int_{t-h_U}^{t-h(t)} f(x(s))ds$ was used, which is different from those in the literature. Also, by taking the states $x(t-h(t))$ and $x(t-h_U)$ as interval of integral terms as shown in the second and third terms of V_3, more information on the cross terms in $(f(x(t-h(t))), x(t-h(t)), \dot{x}(t-h(t)))$ and $(f(x(t-h_U)), x(t-h_U), \dot{x}(t-h_U))$ were utilized, which has not been proposed yet. Furthermore, the term $\int_{t-h_U}^{t-h(t)} \beta^T(s)\mathcal{Q}_2\beta(s)ds$ are chosen as LK functional for the first time when $h_{Dl} \le \dot{h}(t) \le h_{Du}$. These three considerations are main differences in the construction of the LK functional candidate.

Remark 3: Based on the condition (8) and by S-procedure, most of the previous papers were utilized (33) in deriving the asymptotic stability criteria until now. As mentioned in the introduction, all works in [22]–[27] had chosen the delay-partitioning number as two as a tradeoff between computational burden and enhancement of feasible region in stability criteria. That is, the condition $0 \le h(t) \le h_U$ is divided into $0 \le h(t) \le h_U/2$ and $h_U/2 \le h(t) \le h_U$. It should be noted that when the number of delay-partitioning number increases, the matrix formulation becomes more complex and the dimension of stability condition grows bigger because the dimension of an augmented vector increases. In this paper, inspired by the fact that the ability and performance are related to the choice of activation functions [31], the bounding of activation function $k_i^- \le (f_i(u)/u) \le k_i^+$ is divided into two subintervals such as $k_i^- \le (f_i(u)/u) \le (k_i^- + k_i^+)/2$ and $(k_i^- + k_i^+)/2 \le (f_i(u)/u) \le k^+$ instead of using the delay-partitioning approach. This result will be introduced in Theorem 2. Through three numerical examples, it will be shown Theorem 2 significantly improves the feasible region of stability criterion comparing with those of Theorem 1.

Next, based on the results of Theorem 1, a novel approach for delay-dependent stability criterion for (6) is introduced. For the sake of simplicity in matrix representation, the notations for some matrices of Theorem 2 are defined as

$$\begin{aligned}
\Theta_a = &-\left[e_8 - e_1\left(\frac{K_m+K_p}{2}\right)\right] H_1 \left[e_8 - e_1 K_m\right]^T \\
&- \left[e_8 - e_1 K_m\right] H_1 \left[e_8 - e_1\left(\frac{K_m+K_p}{2}\right)\right]^T \\
&- \left[e_9 - e_2\left(\frac{K_m+K_p}{2}\right)\right] H_2 \left[e_9 - e_2 K_m\right]^T \\
&- \left[e_9 - e_2 K_m\right] H_2 \left[e_9 - e_2\left(\frac{K_m+K_p}{2}\right)\right]^T \\
&- \left[e_{10} - e_3\left(\frac{K_m+K_p}{2}\right)\right] H_3 \left[e_{10} - e_3 K_m\right]^T \\
&- \left[e_{10} - e_3 K_m\right] H_3 \left[e_{10} - e_3\left(\frac{K_m+K_p}{2}\right)\right]^T \\
\Theta_b = &-\left[e_8 - e_1 K_p\right] H_4 \left[e_8 - e_1\left(\frac{K_m+K_p}{2}\right)\right]^T \\
&- \left[e_8 - e_1\left(\frac{K_m+K_p}{2}\right)\right] H_4 \left[e_8 - e_1 K_p\right]^T \\
&- \left[e_9 - e_2 K_p\right] H_5 \left[e_9 - e_2\left(\frac{K_m+K_p}{2}\right)\right]^T \\
&- \left[e_9 - e_2\left(\frac{K_m+K_p}{2}\right)\right] H_5 \left[e_9 - e_2 K_p\right]^T \\
&- \left[e_{10} - e_3 K_p\right] H_6 \left[e_{10} - e_3\left(\frac{K_m+K_p}{2}\right)\right]^T \\
&- \left[e_{10} - e_2\left(\frac{K_m+K_p}{2}\right)\right] H_6 \left[e_{10} - e_3 K_p\right]^T . \quad (36)
\end{aligned}$$

Now, the following theorem is the second main result.

Theorem 2: For a given positive scalar h_U, any one h_{Dl} and h_{Du} with C1, diagonal matrices $K_p = \text{diag}\{k_1^+, \ldots, k_n^+\}$ and $K_m = \text{diag}\{k_1^-, \ldots, k_n^-\}$, (6) is asymptotically stable for $0 \le h(t) \le h_U$ and $h_{Dl} \le \dot{h}(t) \le h_{Du} < 1$ if there exist positive diagonal matrices $\Lambda_i = \text{diag}\{\lambda_{1i}, \ldots, \lambda_{ni}\}$ $(i = 1, 2, 3)$, $\Delta_i = \text{diag}\{\delta_{1i}, \ldots, \delta_{ni}\}$ $(i = 1, 2, 3)$, $H_i = \text{diag}\{h_{1i}, \ldots, h_{ni}\}$ $(i = 1, \ldots, 6)$, positive definite matrices $\mathcal{R} \in \mathbb{R}^{5n\times 5n}$, $\mathcal{N} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_1 \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_2 \in \mathbb{R}^{3n\times 3n}$, $\mathcal{G} \in \mathbb{R}^{3n\times 3n}$, $Q_i (i = 3, 4, 5) \in \mathbb{R}^{n\times n}$, and any matrix $\mathcal{S} \in \mathbb{R}^{3n\times 3n}$ and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$, satisfying the following LMIs:

$$\left(\Gamma^{\perp}\right)^T \left(\Sigma_{1[\nabla_d^k]} + \Theta_a\right)\left(\Gamma^{\perp}\right) < 0 \quad (37)$$

$$\left(\Gamma^{\perp}\right)^T \left(\Sigma_{1[\nabla_d^k]} + \Theta_b\right)\left(\Gamma^{\perp}\right) < 0 \quad (38)$$

$$\begin{bmatrix} \mathcal{G} & \mathcal{S} \\ \star & \mathcal{G} \end{bmatrix} > 0 \quad (39)$$

$$\begin{bmatrix} Q_4 & P_1 \\ \star & Q_5 \end{bmatrix} > 0, \qquad \begin{bmatrix} Q_4 & P_2 \\ \star & Q_5 \end{bmatrix} > 0 \quad \forall k = 1, 2 \quad (40)$$

where $\Sigma_{1[\nabla_d^k]}$, and Γ are defined in (15), Θ_a and Θ_b are in (36), and $\Gamma^{\perp}$ is the right orthogonal complement of Γ.

Proof: For positive diagonal matrices Λ_i, Δ_i $(i = 1, 2, 3)$ and positive definite matrices $\mathcal{R}$, $\mathcal{N}$, $\mathcal{G}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $Q_i (i = 3, 4, 5)$, let us consider the same LK functional (62) proposed in Theorem 1.

Case 1:

$$k_i^- \le (f_i(u) - f_i(v)/u - v) \le (k_i^- + k_i^+)/2.$$

Let us choose $v = 0$. It should be noted that the condition $k_i^- \le (f_i(u)/u) \le (k_i^- + k_i^+)/2$ is equivalent to

$$\begin{aligned} &\left[f_i(u) - k_i^- u\right]\left[f_i(u) - ((k_i^- + k_i^+)/2)u\right] < 0, \\ &i = 1, \ldots, n. \end{aligned} \quad (41)$$

From (41), for any positive diagonal matrices $H_1 = \text{diag}\{h_{11}, \ldots, h_{1n}\}$, $H_2 = \text{diag}\{h_{21}, \ldots, h_{2n}\}$, and $H_3 = \text{diag}$

$\{h_{31}, \ldots, h_{3n}\}$, the following inequality holds:

$$\begin{aligned}
0 \le & -2\sum_{i=1}^{n} h_{1i}\left[f_i(x_i(t)) - k_i^- x_i(t)\right] \\
&\times \left[f_i(x_i(t)) - \left(\frac{k_i^- + k_i^+}{2}\right)x_i(t)\right] \\
&-2\sum_{i=1}^{n} h_{2i}\left[f_i(x_i(t-h(t))) - k_i^- x_i(t-h(t))\right] \\
&\times \left[f_i(x_i(t-h(t))) - \left(\frac{k_i^- + k_i^+}{2}\right)x_i(t-h(t))\right] \\
&-2\sum_{i=1}^{n} h_{3i}\left[f_i(x_i(t-h_U)) - k_i^- x_i(t-h_U)\right] \\
&\times \left[f_i(x_i(t-h_U)) - \left(\frac{k_i^- + k_i^+}{2}\right)x_i(t-h_U)\right] \\
= & \zeta^T(t)\Theta_a\zeta(t). \qquad (42)
\end{aligned}$$

Then, from the proof of Theorem 1, when $k_i^- \le (f_i(u)/u) \le (k_i^- + k_i^+)/2$, an upper bound of $\dot{V}$ can be

$$\dot{V} \le \zeta^T(t)\left\{\Sigma_{1[\dot{h}(t)]} + \Theta_a\right\}\zeta(t) \qquad (43)$$

with $0 = \Gamma\zeta(t)$. Therefore, from Lemma 2 and S-procedure [37], if (37), (39), and (40) hold, then (6) is asymptotically stable for $0 \le h(t) \le h_U$, $h_{Dl} \le \dot{h}(t) \le h_{Du} < 1$, and $k_i^- \le (f_i(u)/u) \le (k_i^- + k_i^+)/2$.

Case 2:

$$(k_i^- + k_i^+)/2 \le (f_i(u) - f_i(v)/u - v) \le k_i^+.$$

Let us choose $v = 0$. It should be noted that the condition $(k_i^- + k_i^+)/2 \le (f_i(u)/u) \le k_i^+$ is equivalent to

$$\begin{aligned}
&\left[f_i(u) - \left(\frac{k_i^- + k_i^+}{2}\right)u\right]\left[f_i(u) - k_i^+ u\right] < 0, \\
&i = 1, \ldots, n. \qquad (44)
\end{aligned}$$

From (44), for any positive diagonal matrices $H_4 = \mathrm{diag}\{h_{41}, \ldots, h_{4n}\}$, $H_5 = \mathrm{diag}\{h_{51}, \ldots, h_{5n}\}$, and $H_6 = \mathrm{diag}\{h_{61}, \ldots, h_{6n}\}$, the following inequality holds:

$$0 \le \zeta^T(t)\Theta_b\zeta(t). \qquad (45)$$

Then, from the proof of Theorem 1, when $(k_i^- + k_i^+)/2 \le (f_i(u)/u) \le k_i^+$, an upper bound of $\dot{V}$ can be

$$\dot{V} \le \zeta^T(t)\left\{\Sigma_{1[\dot{h}(t)]} + \Theta_b\right\}\zeta(t) \qquad (46)$$

with $0 = \Gamma\zeta(t)$.

Therefore, from Lemma 2 and S-procedure [37], if (38)–(40) hold, then (6) is asymptotically stable for $0 \le h(t) \le h_U$, $h_{Dl} \le \dot{h}(t) \le h_{Du} < 1$, and $(k_i^- + k_i^+)/2 \le (f_i(u)/u) \le k_i^+$. Thus, the feasibility of (37)–(40) means that (6) is asymptotically stable for $0 \le h(t) \le h_U$, $h_{Dl} \le \dot{h}(t) \le h_{Du} < 1$, and $k_i^- \le (f_i(u)/u) \le k_i^+$. This completes the proof of Theorem 2. ■

Remark 4: As mentioned in [15], the activation functions of the transformed system (6) also satisfy the condition (7). In Theorem 3, by choosing (u, v) in (7) as $(x(t), x(t-h(t)))$, and $(x(t-h(t)), x(t-h_U))$ at each subintervals $k_i^- \le (f_i(u)/u) \le (k_i^- + k_i^+)/2$ and $(k_i^- + k_i^+)/2 \le (f_i(u)/u) \le k^+$, respectively, more information on cross terms among the states $f(x(t)), f(x(t-h(t))), f(x(t-h_U)), x(t), x(t-h(t))$, and $x(t-h_U)$ will be utilized, which may lead to less conservative stability criteria. This idea has not been considered earlier in the literature. Through three numerical examples utilized in the literature, it will be shown that the newly proposed activation condition significantly enhances the feasible region of stability criterion by comparing maximum delay bounds with the results obtained by Theorem 2.

The following matrix notations will be used in Theorem 3 for the sake of simplicity:

$$\begin{aligned}
\Omega_a = & -\left[e_8 - e_9 - (e_1 - e_2)K_m\right]H_7 \\
&\times \left[e_8 - e_9 - (e_1 - e_2)\left(\frac{K_m + K_p}{2}\right)\right]^T \\
&- \left[e_8 - e_9 - (e_1 - e_2)\left(\frac{K_m + K_p}{2}\right)\right]H_7 \\
&\times \left[e_8 - e_9 - (e_1 - e_2)K_m\right]^T \\
&- \left[e_9 - e_{10} - (e_2 - e_3)K_m\right]H_8 \times \\
&\left[e_9 - e_{10} - (e_2 - e_3)\left(\frac{K_m + K_p}{2}\right)\right]^T \\
&- \left[e_9 - e_{10} - (e_2 - e_3)\left(\frac{K_m + K_p}{2}\right)\right]H_8 \\
&\times \left[e_9 - e_{10} - (e_2 - e_3)K_m\right]^T \\
\Omega_b = & -\left[e_8 - e_9 - (e_1 - e_2)\left(\frac{K_m + K_p}{2}\right)\right]H_9 \\
&\times \left[e_8 - e_9 - (e_1 - e_2)K_p\right]^T \\
&- \left[e_8 - e_9 - (e_1 - e_2)K_p\right]H_9 \times \\
&\left[e_8 - e_9 - (e_1 - e_2)\left(\frac{K_m + K_p}{2}\right)\right]^T \\
&- \left[e_9 - e_{10} - (e_2 - e_3)\left(\frac{K_m + K_p}{2}\right)\right]H_{10} \\
&\times \left[e_9 - e_{10} - (e_2 - e_3)K_p\right]^T \\
&- \left[e_9 - e_{10} - (e_2 - e_3)K_p\right]H_{10} \\
&\times \left[e_9 - e_{10} - (e_2 - e_3)\left(\frac{K_m + K_p}{2}\right)\right]^T. \qquad (47)
\end{aligned}$$

Now, the following theorem is the final main result.

Theorem 3: For a given positive scalar h_U, any ones h_{Dl} and h_{Du} with C1, diagonal matrices $K_p = \mathrm{diag}\{k_1^+, \ldots, k_n^+\}$ and $K_m = \mathrm{diag}\{k_1^-, \ldots, k_n^-\}$, (6) is asymptotically stable for $0 \le h(t) \le h_U$ and $h_{Dl} \le \dot{h}(t) \le h_{Du} < 1$ if there exist positive diagonal matrices $\Lambda_i = \mathrm{diag}\{\lambda_{1i}, \ldots, \lambda_{ni}\}$ $(i = 1, 2, 3)$, $\Delta_i = \mathrm{diag}\{\delta_{1i}, \ldots, \delta_{ni}\}$ $(i = 1, 2, 3)$, $H_i = \mathrm{diag}\{h_{1i}, \ldots, h_{ni}\}(i = 1, \ldots, 10)$, positive definite matrices $\mathcal{R} \in \mathbb{R}^{5n\times5n}$, $\mathcal{N} \in \mathbb{R}^{3n\times3n}$, $\mathcal{Q}_1 \in \mathbb{R}^{3n\times3n}$, $\mathcal{Q}_2 \in \mathbb{R}^{3n\times3n}$, $\mathcal{G} \in \mathbb{R}^{3n\times3n}$, $Q_i(i = 3, 4, 5) \in \mathbb{R}^{n\times n}$, and any matrix $\mathcal{S} \in \mathbb{R}^{3n\times3n}$ and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$, satisfying the following LMIs:

$$\left(\Gamma^{\perp}\right)^T \left(\Sigma_{1[\nabla_d^k]} + \Theta_a + \Omega_a\right)\left(\Gamma^{\perp}\right) < 0 \tag{48}$$

$$\left(\Gamma^{\perp}\right)^T \left(\Sigma_{1[\nabla_d^k]} + \Theta_b + \Omega_b\right)\left(\Gamma^{\perp}\right) < 0 \tag{49}$$

$$\begin{bmatrix} \mathcal{G} & \mathcal{S} \\ \star & \mathcal{G} \end{bmatrix} > 0 \tag{50}$$

$$\begin{bmatrix} Q_4 & P_1 \\ \star & Q_5 \end{bmatrix} > 0, \quad \begin{bmatrix} Q_4 & P_2 \\ \star & Q_5 \end{bmatrix} > 0 \quad \forall k = 1, 2 \tag{51}$$

where $\Sigma_{1[\nabla_d^k]}$, and Γ are defined in (15), Θ_a and Θ_b are in (36), Ω_a and Ω_b are in (47), and $\Gamma^{\perp}$ is the right orthogonal complement of Γ.

Proof:

Case 1:

$$k_i^- \le (f_i(u) - f_i(v)/u - v) \le (k_i^- + k_i^+)/2.$$

For Case 1, the following conditions hold:

$$\begin{aligned} k_i^- &\le \frac{f_i(x_i(t)) - f_i(x_i(t-h(t)))}{x_i(t) - x_i(t-h(t))} \le (k_i^- + k_i^+)/2 \\ k_i^- &\le \frac{f_i(x_i(t-h(t))) - f_j(x_i(t-h_U))}{x_i(t-h(t)) - x_i(t-h_U)} \le (k_i^- + k_i^+)/2 \\ i &= 1, \ldots, n. \end{aligned} \tag{52}$$

For $i = 1, \ldots, n$, the above two conditions are equivalent to

$$\begin{aligned} &\left[f_i(x_i(t)) - f_i(x_i(t-h(t)) - k_i^-(x_i(t) - x_i(t-h(t)))\right] \\ &\times\left[f_i(x_i(t)) - f_i(x_i(t-h(t))) - \left(\frac{k_i^- + k_i^+}{2}\right)\right. \\ &\quad \left.\times (x_i(t) - x_i(t-h(t)))\right] \le 0 \end{aligned} \tag{53}$$

$$\begin{aligned} &\left[f_i(x_i(t-h(t))) - f_i(x_i(t-h_U))\right. \\ &\left.-k_i^-(x_i(t-h(t)) - x_i(t-h_U))\right] \\ &\times [f_i(x_i(t-h(t))) - f_i(x_i(t-h_U)) - \left(\frac{k_i^- + k_i^+}{2}\right) \\ &\quad \times (x_i(t-h(t)) - x_i(t-h_U))] \le 0. \end{aligned} \tag{54}$$

Therefore, for any positive diagonal matrices $H_7 = \text{diag}\{h_{7i}, \ldots, h_{7n}\}$, and $H_8 = \text{diag}\{h_{8i}, \ldots, h_{8n}\}$, the following inequality is satisfied:

$$\begin{aligned} 0 \le\ & -2\sum_{i=1}^{n}\left\{h_{7i}[f_i(x_i(t)) - f_i(x_i(t-h(t)))\right. \\ &\qquad -k_i^-(x_i(t) - x_i(t-h(t)))] \\ &\qquad \times [f_i(x_i(t)) - f_i(x_i(t-h(t))) \\ &\qquad \left.-\left(\frac{k_i^- + k_i^+}{2}\right)(x_i(t) - x_i(t-h(t)))]\right\} \\ & -2\sum_{i=1}^{n}\left\{h_{8i}[f_i(x_i(t-h(t))) - f_i(x_i(t-h_U))\right. \\ &\qquad -k_i^-(x_i(t-h(t)) - x_i(t-h_U))] \\ &\qquad \times[f_i(x_i(t-h(t))) - f_i(x_i(t-h_U)) \\ &\qquad \left.-\left(\frac{k_i^- + k_i^+}{2}\right)(x_i(t-h(t)) - x_i(t-h_U))]\right\} \\ =\ & \zeta^T(t)\Omega_a\zeta(t). \end{aligned} \tag{55}$$

By considering (55) in Case I of Theorem 2, (48) can be obtained.

Case 2:

$$(k_i^- + k_i^+)/2 \le (f_i(u) - f_i(v)/u - v) \le k_i^+.$$

For this case, using the similar method introduced in case 1 of Theorem 3, it can be easily checked that

$$0 \le \zeta^T(t)\Omega_b\zeta(t) \tag{56}$$

holds. Thus, by considering (56) in case 2 of Theorem 2, (49) can be obtained. This completes our proof. ■

Remark 5: When $\dot{h}(t) \le h_D$, the state $\dot{x}(t-h(t))$ cannot be utilized as augmented vector $\zeta(t)$ by the methods presented in the proofs of Theorems 1–3. Thus, V_1 utilized in Theorems 1–3 should be modified. Also, the second term of proposed LK functional V_3 cannot be utilized since the term $\Phi_{2[\dot{h}(t)]}$ cannot be estimated with the constraint $\dot{h}(t) \le h_D$. With these considerations and based on the result of Theorem 3, the corresponding stability criterion for C2 will be introduced as Corollary 1.

In Corollary 1, block entry matrices $\widetilde{e}_i(t) \in \mathbb{R}^{12n\times n}$ will be used and the following notations are defined for the sake of simplicity of matrix notation:

$$\begin{aligned} \widetilde{\zeta}^T(t) &= \Big[x^T(t) \ \ x^T(t-h(t)) \ \ x^T(t-h_U) \ \ \dot{x}^T(t) \\ &\qquad \times \dot{x}^T(t-h_U) \ \ \int_{t-h(t)}^{t} x^T(s)ds \\ &\qquad \times \int_{t-h_U}^{t-h(t)} x^T(s)ds \ \ f^T(x(t)) \ \ f^T(x(t-h(t))) \\ &\qquad \times f^T(x(t-h_U)) \ \ \int_{t-h(t)}^{t} f^T(x(s))ds \\ &\qquad \times \int_{t-h_U}^{t-h(t)} f^T(x(s))ds\Big] \\ \widetilde{\Gamma} &= \left[-A \ 0 \ 0 \ -I \ 0 \ 0 \ 0 \ W_0 \ W_1 \ 0 \ 0 \ 0\right] \\ \widetilde{\alpha}^T(t) &= \left[x^T(t) \ x^T(t-h_U) \ \textstyle\int_{t-h_U}^{t} x^T(s)ds \ \int_{t-h_U}^{t} f^T(x(s))ds\right. \\ \widetilde{\Pi}_1 &= \left[\widetilde{e}_1 \ \widetilde{e}_3 \ \widetilde{e}_6 + \widetilde{e}_7 \ \widetilde{e}_{11} + \widetilde{e}_{12}\right] \\ \widetilde{\Pi}_2 &= \left[\widetilde{e}_4 \ \widetilde{e}_5 \ \widetilde{e}_1 - \widetilde{e}_3 \ \widetilde{e}_8 - \widetilde{e}_{10}\right] \\ \widetilde{\Pi}_3 &= \left[\widetilde{e}_1 \ \widetilde{e}_4 \ \widetilde{e}_8\right], \quad \widetilde{\Pi}_4 = \left[\widetilde{e}_3 \ \widetilde{e}_5 \ \widetilde{e}_{10}\right] \\ \widetilde{\Pi}_6 &= \left[\widetilde{e}_6 \ \widetilde{e}_1 - \widetilde{e}_2 \ \widetilde{e}_{11} \ \widetilde{e}_7 \ \widetilde{e}_2 - \widetilde{e}_3 \ \widetilde{e}_{12}\right] \\ \widetilde{\Pi}_a &= \left[\widetilde{e}_1 \ \widetilde{e}_2\right], \quad \widetilde{\Pi}_b = \left[\widetilde{e}_2 \ \widetilde{e}_9\right] \\ \widetilde{\Phi}_1 &= [\widetilde{e}_8 - \widetilde{e}_1 K_m]\Lambda_1\widetilde{e}_4^T + \widetilde{e}_4\Lambda_1[\widetilde{e}_8 - \widetilde{e}_1 K_m]^T \\ &\quad + \left[\widetilde{e}_1 K_p - \widetilde{e}_8\right]\Delta_1\widetilde{e}_4^T + \widetilde{e}_4\Delta_1\left[\widetilde{e}_1 K_p - \widetilde{e}_8\right]^T \\ &\quad + [\widetilde{e}_{10} - \widetilde{e}_3 K_m]\Lambda_3\widetilde{e}_5^T + \widetilde{e}_5\Lambda_3[\widetilde{e}_{10} - \widetilde{e}_3 K_m]^T \\ &\quad + \left[\widetilde{e}_3 K_p - \widetilde{e}_{10}\right]\Delta_3\widetilde{e}_5^T + \widetilde{e}_5\Delta_3\left[\widetilde{e}_3 K_p - \widetilde{e}_{10}\right]^T \\ \widetilde{\Xi} &= (h_U^2/2)^2\widetilde{e}_4 Q_3\widetilde{e}_4^T \\ &\quad -(h_U\widetilde{e}_1 - \widetilde{e}_6 - \widetilde{e}_7)Q_3(h_U\widetilde{e}_1 - \widetilde{e}_6 - \widetilde{e}_7)^T \\ \widetilde{\Psi} &= h_U\widetilde{e}_1 Q_4\widetilde{e}_1^T + h_U\widetilde{e}_4 Q_5\widetilde{e}_4^T + \widetilde{e}_1 P_1\widetilde{e}_1^T \\ &\quad +\widetilde{e}_2(-P_1 + P_2)\widetilde{e}_2^T - \widetilde{e}_3 P_2\widetilde{e}_3^T \\ \widetilde{\Im} &= \widetilde{\Pi}_a\mathcal{Q}_1\widetilde{\Pi}_a^T - (1 - h_{Du})\widetilde{\Pi}_b\mathcal{Q}_1\widetilde{\Pi}_b^T \end{aligned}$$

$$\Sigma_2 = \widetilde{\Pi}_1\mathcal{R}\widetilde{\Pi}_2^T + \widetilde{\Pi}_2\mathcal{R}\widetilde{\Pi}_1^T + \widetilde{\Pi}_3\mathcal{N}\widetilde{\Pi}_3^T - \widetilde{\Pi}_4\mathcal{N}\widetilde{\Pi}_4^T + \widetilde{\Phi}_1 + h_U^2\widetilde{\Pi}_3\mathcal{G}\widetilde{\Pi}_3^T - \widetilde{\Pi}_6\begin{bmatrix}\mathcal{G} & \mathcal{S}\\ \star & \mathcal{G}\end{bmatrix}\Pi_6^T + \widetilde{\Xi} + \widetilde{\Psi}. \tag{57}$$

Corollary 1: For a given positive scalar h_U and $h_D u$ with C2, diagonal matrices $K_p = \text{diag}\{k_1^+,\ldots,k_n^+\}$, and $K_m = \text{diag}\{k_1^-,\ldots,k_n^-\}$, (6) is asymptotically stable for $0 \le h(t) \le h_U$ and $\dot{h}(t) \le h_U$ if there exist positive diagonal matrices $\Lambda_i = \text{diag}\{\lambda_{1i},\ldots,\lambda_{ni}\}$ $(i = 1,3)$, $\Delta_i = \text{diag}\{\delta_{1i},\ldots,\delta_{ni}\}$ $(i = 1,3)$, $H_i = \text{diag}\{h_{1i},\ldots,h_{ni}\}$ $(i = 1,\ldots,10)$, positive definite matrices $\mathcal{R} \in \mathbb{R}^{5n\times 5n}$, $\mathcal{N} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{G} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{Q}_1 \in \mathbb{R}^{2n\times 2n}$ $Q_i (i = 3,4,5) \in \mathbb{R}^{n\times n}$, and any matrix $\mathcal{S} \in \mathbb{R}^{3n\times 3n}$ and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1,2)$, satisfying the following LMIs:

$$\left(\widetilde{\Gamma}^\perp\right)^T\left(\Sigma_2 + \widetilde{\Theta}_a + \widetilde{\Omega}_a + \widetilde{\Im}\right)\left(\widetilde{\Gamma}^\perp\right) < 0 \tag{58}$$

$$\left(\widetilde{\Gamma}^\perp\right)^T\left(\Sigma_2 + \widetilde{\Theta}_b + \widetilde{\Omega}_b + \widetilde{\Im}\right)\left(\widetilde{\Gamma}^\perp\right) < 0 \tag{59}$$

$$\begin{bmatrix}\mathcal{G} & \mathcal{S}\\ \star & \mathcal{G}\end{bmatrix} > 0 \tag{60}$$

$$\begin{bmatrix}Q_4 & P_1\\ \star & Q_5\end{bmatrix} > 0, \qquad \begin{bmatrix}Q_4 & P_2\\ \star & Q_5\end{bmatrix} > 0 \tag{61}$$

where Σ_2, $\widetilde{\Gamma}$, are defined in (57), $\widetilde{\Gamma}^\perp$ is the right orthogonal complement of $\widetilde{\Gamma}$, and $\widetilde{\Theta}_a$, $\widetilde{\Theta}_b$, $\widetilde{\Omega}_a$, and $\widetilde{\Omega}_b$ have the same notations defined in (36) and (47) with the block entry matrices $\widetilde{e}_i(t) \in \mathbb{R}^{12n\times n}$ $(i = 1,\ldots,12)$.

Proof: For positive diagonal matrices Λ_i, Δ_i $(i = 1,3)$ and positive definite matrices $\mathcal{R}$, $\mathcal{N}$,$\mathcal{G}$, and $Q_i (i = 3,4,5)$, let us take the LK functional candidate

$$V = \sum_{i=1}^{7} V_i \tag{62}$$

where

$$V_1 = \widetilde{\alpha}^T(t)\mathcal{R}\widetilde{\alpha}(t)$$

$$V_2 = \int_{t-h_U}^{t}\beta^T(s)\mathcal{N}\beta(s)ds$$

$$V_3 = 2\sum_{i=1}^{n}\left(\lambda_{1i}\int_0^{x_i(t)}(f_i(s) - k_i^- s)ds + \delta_{1i}\int_0^{x_i(t)}(k_i^+ s - f_i(s))ds\right) + 2\sum_{i=1}^{n}\left(\lambda_{3i}\int_0^{x_i(t-h_U)}(f_i(s) - k_i^- s)ds + \delta_{3i}\int_0^{x_i(t-h_U)}(k_i^+ s - f_i(s))ds\right)$$

$$V_4 = \int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\ f(x(s))\end{bmatrix}^T\mathcal{Q}_1\begin{bmatrix}x(s)\\ f(x(s))\end{bmatrix}ds$$

$$V_5 = h_U\int_{t-h_U}^{t}\int_s^t\beta^T(u)\mathcal{G}\beta(u)duds$$

$$V_6 = (h_U^2/2)\int_{t-h_U}^{t}\int_s^t\int_u^t\dot{x}^T(v)Q_3\dot{x}(v)dvduds$$

$$V_7 = \int_{t-h_U}^{t}\int_s^t x^T(u)Q_4x(u)duds + \int_{t-h_U}^{t}\int_s^t\dot{x}^T(u)Q_5\dot{x}(u)duds \tag{63}$$

and $\widetilde{\alpha}(t)$ are defined in (57) and $\beta(t)$ are in (15).

With the augmented vector $\widetilde{\zeta}(t)$ defined in (57) and based on the proof of Theorem 3, one can easily check that (58)–(61) guarantee the asymptotic stability for (6). ■

Finally, based on the result of Corollary 1, when information about the upper bound of $\dot{h}(t)$ is unknown, the corresponding stability criterion will be described as Corollary 2 by choosing $\mathcal{Q}_1 = 0$.

Corollary 2: For a given positive scalar h_U with C3, diagonal matrices $K_p = \text{diag}\{k_1^+,\ldots,k_n^+\}$, and $K_m = \text{diag}\{k_1^-,\ldots,k_n^-\}$, (6) is asymptotically stable for $0 \le h(t) \le h_U$ if there exist positive diagonal matrices $\Lambda_i = \text{diag}\{\lambda_{1i},\ldots,\lambda_{ni}\}$ $(i = 1,3)$, $\Delta_i = \text{diag}\{\delta_{1i},\ldots,\delta_{ni}\}$ $(i = 1,3)$, $H_i = \text{diag}\{h_{1i},\ldots,h_{ni}\}$ $(i = 1,\ldots,10)$, positive definite matrices $\mathcal{R} \in \mathbb{R}^{5n\times 5n}$, $\mathcal{N} \in \mathbb{R}^{3n\times 3n}$, $\mathcal{G} \in \mathbb{R}^{3n\times 3n}$, $Q_i (i = 3,4,5) \in \mathbb{R}^{n\times n}$, and any matrix $\mathcal{S} \in \mathbb{R}^{3n\times 3n}$ and symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1,2)$, satisfying the following LMIs:

$$\left(\widetilde{\Gamma}^\perp\right)^T\left(\Sigma_2 + \widetilde{\Theta}_a + \widetilde{\Omega}_a\right)\left(\widetilde{\Gamma}^\perp\right) < 0 \tag{64}$$

$$\left(\widetilde{\Gamma}^\perp\right)^T\left(\Sigma_2 + \widetilde{\Theta}_b + \widetilde{\Omega}_b\right)\left(\widetilde{\Gamma}^\perp\right) < 0 \tag{65}$$

$$\begin{bmatrix}\mathcal{G} & \mathcal{S}\\ \star & \mathcal{G}\end{bmatrix} > 0 \tag{66}$$

$$\begin{bmatrix}Q_4 & P_1\\ \star & Q_5\end{bmatrix} > 0, \qquad \begin{bmatrix}Q_4 & P_2\\ \star & Q_5\end{bmatrix} > 0 \tag{67}$$

where all the notations of (64)–(67) are the same as in Corollary 1.

IV. Numerical Examples

In this section, three numerical examples will be used to check the feasibility and improvement of the stability criteria.

Example 1: Consider the neural networks (6) with the parameters

$$A = \begin{bmatrix}1.5 & 0\\ 0 & 0.7\end{bmatrix}, \quad W_0 = \begin{bmatrix}0.0503 & 0.0454\\ 0.0987 & 0.2075\end{bmatrix}$$
$$W_1 = \begin{bmatrix}0.2381 & 0.9320\\ 0.0388 & 0.5062\end{bmatrix}$$
$$K_p = \text{diag}\{0.3,\ 0.8\},\ K_m = \text{diag}\{0,\ 0\}. \tag{68}$$

With the condition $-h_D \le \dot{h}(t) \le h_D$, our results obtained by Theorems 1–3 to the above system are shown in Table I.

TABLE I
DELAY BOUNDS h_U WITH DIFFERENT h_D (EXAMPLE 1)

Method	Condition of $\dot{h}(t)$	h_D			
		0.4	0.45	0.5	0.55
[24] $(m=2)$	$0 \le \dot{h}(t) \le h_D$	4.39	3.67	3.46	3.41
Theorem 2 [20]	$-h_D \le \dot{h}(t) \le h_D$	4.8401	4.0626	3.8083	3.7064
[25] $(m=2)$	$0 \le \dot{h}(t) \le h_D$	5.2420	4.4301	4.1055	3.9231
Theorem 1	$-h_D \le \dot{h}(t) \le h_D$	5.0588	4.2603	4.0604	4.0185
Theorem 2	$-h_D \le \dot{h}(t) \le h_D$	5.3079	4.5267	4.2924	4.1903
Theorem 3	$-h_D \le \dot{h}(t) \le h_D$	9.7094	7.7523	6.8570	6.2977
Corollary 1	$\dot{h}(t) \le h_D$	4.8748	4.2702	4.0551	3.9369

$*$ m is delay-partitioning number

TABLE II
DELAY BOUNDS h_U WITH DIFFERENT h_D (EXAMPLE 2)

Method	Condition of $\dot{h}(t)$	h_D		
		0.8	0.9	unknown or ≥ 1
Theorem 2 [28] ($\rho = 0.8$)	$0 \leq \dot{h}(t) \leq h_D$	2.5406	1.7273	1.5161
Theorem 1 [22] ($m = 2$)	$\dot{h}(t) \leq h_D$	2.8654	1.9508	–
Corollary 1 [22] ($m = 2$)	–	–	–	1.7809
Theorem 1 [23] ($m = 2$)	$\dot{h}(t) \leq h_D$	2.8854	1.9631	–
Theorem 1 without V_2 [23] ($m = 2$)	–	–	–	1.7810
Theorem 1 [25] ($m = 2$)	$0 \leq \dot{h}(t) \leq h_D$	3.0604	1.9956	–
Corollary 1 [25] ($m = 2$)	–	–	–	1.7860
Theorem 1 [27] ($m = 2$)	$\dot{h}(t) \leq h_D$	3.0640	2.0797	–
Corollary 1 [27] ($m = 2$)	–	–	–	1.9207
Theorem 1	$-h_D \leq \dot{h}(t) \leq h_D$	5.4714	3.7440	–
Theorem 2	$-h_D \leq \dot{h}(t) \leq h_D$	6.5848	4.1767	–
Theorem 3	$-h_D \leq \dot{h}(t) \leq h_D$	7.5173	5.3993	–
Corollary 1	$\dot{h}(t) \leq h_D$	3.7236	2.9229	–
Corollary 2	–	–	–	2.9208

$*$ m is delay-partitioning number

TABLE III
COMPARISON OF DELAY BOUNDS h_U WITH THE RESULTS OF [29] FOR DIFFERENT h_D (EXAMPLE 2)

Method	Condition of $\dot{h}(t)$	h_D		
		0.8	0.9	unknown
Theorem 1 [29] ($m = 1$)	$0 \leq \dot{h}(t) \leq h_D$	3.6456	2.3361	–
Theorem 1 [29] ($m = 1$, $P_{12} = P_{22} = 0$, $R = 0$)	–	–	–	1.4916
Theorem 1 [29] ($m = 2$)	$0 \leq \dot{h}(t) \leq h_D$	4.6752	3.0208	–
Theorem 1 [29] ($m = 2$, $P_{12} = P_{22} = 0$, $R = 0$)	–	–	–	1.7810
Theorem 1 [29] ($m = 3$)	$0 \leq \dot{h}(t) \leq h_D$	5.3523	3.4668	–
Theorem 1 [29] ($m = 3$, $P_{12} = P_{22} = 0$, $R = 0$)	–	–	–	1.9645
Theorem 1 [29] ($m = 4$)	$0 \leq \dot{h}(t) \leq h_D$	5.7957	3.7639	–
Theorem 1 [29] ($m = 4$, $P_{12} = P_{22} = 0$, $R = 0$)	–	–	–	2.0727
Theorem 1 [29] ($m = 5$)	$0 \leq \dot{h}(t) \leq h_D$	6.1032	3.9696	–
Theorem 1 [29] ($m = 5$, $P_{12} = P_{22} = 0$, $R = 0$)	–	–	–	2.1445
Theorem 1	$0 \leq \dot{h}(t) \leq h_D$	5.9656	4.0364	–
Theorem 2	$0 \leq \dot{h}(t) \leq h_D$	7.4425	4.6195	–
Theorem 3	$0 \leq \dot{h}(t) \leq h_D$	8.6008	5.9978	–
Corollary 2	–	–	–	2.9208

$*$ m is delay-partitioning number

TABLE IV
DELAY BOUNDS h_U WITH DIFFERENT h_D (EXAMPLE 3, CASE 1)

Method	Condition of $\dot{h}(t)$	h_D			
		0.1	0.5	0.9	unknown
Theorem 2 [28] ($\rho = 0.6$)	$0 \leq \dot{h}(t) \leq h_D$	3.3574	2.5915	2.1306	2.0779
Proposition 2 [26] ($m = 2$)	$\dot{h}(t) \leq h_D$	3.5546	2.6438	2.1349	–
Theorem 1 [23] ($m = 2$)	$\dot{h}(t) \leq h_D$	3.7525	2.7353	2.2760	–
Theorem 1 without V_2 [23] ($m = 2$)	–	–	–	–	2.1326
Theorem 1 [24] ($m = 2$)	$0 \leq \dot{h}(t) \leq h_D$	3.91	2.79	2.33	–
Theorem 1 without Y_{ij} [24] ($m = 2$)	–	–	–	–	2.2047
Theorem 2 [20]	$-h_D \leq \dot{h}(t) \leq h_D$	3.7854	3.2229	2.6422	–
Corollary 1 [20]	–	–	–	–	2.1950
Theorem 1	$-h_D \leq \dot{h}(t) \leq h_D$	3.9269	3.4072	2.8337	–
Theorem 2	$-h_D \leq \dot{h}(t) \leq h_D$	3.9332	3.5277	3.2025	–
Theorem 3	$-h_D \leq \dot{h}(t) \leq h_D$	3.9337	3.5307	3.2627	–
Corollary 1	$\dot{h}(t) \leq h_D$	3.8102	3.1518	2.8402	–
Corollary 2	–	–	–	–	2.8379

$*$ m is delay-partitioning number

Also, when $\dot{h}(t) \leq h_D$, the corresponding results obtained by Corollary 1 are also included in Table I. In the table, the recent results of [20], [24], and [25] are compared with ours. From Table I, it can be seen that Theorem 1 improves the feasible region of stability criteria compared to those of [20] and [24] but falls short compared to the results of [25].

TABLE V

COMPARISON OF DELAY BOUNDS h_U WITH THE RESULTS OF [29] FOR DIFFERENT h_D (EXAMPLE 3, CASE 1)

Method	Condition of $\dot{h}(t)$	h_D				
		0	0.1	0.5	0.9	unknown
Theorem 1 [29] ($m=1$)	$0 \le \dot{h}(t) \le h_D$	3.9376	3.8682	3.5770	2.9124	–
Theorem 1 [29] ($m=1$, $P_{12}=P_{22}=0$, $R=0$)	–	–	–	–	–	2.0094
Theorem 1 [29] ($m=2$)	$0 \le \dot{h}(t) \le h_D$	4.5812	4.4288	4.0089	3.2900	–
Theorem 1 [29] ($m=2$, $P_{12}=P_{22}=0$, $R=0$)	–	–	–	–	–	1.7810
Theorem 1 [29] ($m=3$)	$0 \le \dot{h}(t) \le h_D$	4.6971	4.5328	4.0983	3.3724	–
Theorem 1 [29] ($m=3$, $P_{12}=P_{22}=0$, $R=0$)	–	–	–	–	–	2.2266
Theorem 1 [29] ($m=4$)	$0 \le \dot{h}(t) \le h_D$	4.7400	4.5724	4.1343	3.4066	–
Theorem 1 [29] ($m=4$, $P_{12}=P_{22}=0$, $R=0$)	–	–	–	–	–	2.2415
Theorem 1 [29] ($m=5$)	$0 \le \dot{h}(t) \le h_D$	4.7591	4.5906	4.1518	3.4186	–
Theorem 1 [29] ($m=5$, $P_{12}=P_{22}=0$, $R=0$)	–	–	–	–	–	2.2455
Theorem 1	$0 \le \dot{h}(t) \le h_D$	4.1731	4.0551	3.9308	3.3228	–
Theorem 2	$0 \le \dot{h}(t) \le h_D$	4.1840	4.1089	4.0384	3.7175	–
Theorem 3	$0 \le \dot{h}(t) \le h_D$	4.1844	4.1135	4.0464	3.7768	–
Corollary 2	–	–	–	–	–	2.9208

$*$ m is the delay-partitioning number

TABLE VI

DELAY BOUNDS h_U WITH DIFFERENT h_D (EXAMPLE 3, CASE 2)

Method	Condition of $\dot{h}(t)$	h_D				
		0	0.1	0.5	0.9	unknown
Theorem 1 [24] ($m=2$)	$0 \le \dot{h}(t) \le h_D$	1.9676	1.4673	infeasible	infeasible	–
Theorem 1 without Y_{ij} [24] ($m=2$)	–	–	–	–	–	infeasible
Theorem 2 [20]	$-h_D \le \dot{h}(t) \le h_D$	2.8631	2.4707	infeasible	infeasible	–
Corollary 1 [20]	–	–	–	–	–	infeasible
Theorem 1	$-h_D \le \dot{h}(t) \le h_D$	2.9484	2.6305	0.6716	infeasible	–
Theorem 2	$-h_D \le \dot{h}(t) \le h_D$	3.9862	3.7335	2.7973	1.5598	–
Theorem 3	$-h_D \le \dot{h}(t) \le h_D$	4.0229	3.7824	2.9955	2.1655	–
Corollary 1	$\dot{h}(t) \le h_D$	3.7752	3.4784	2.6390	2.0649	–
Corollary 2	–	–	–	–	–	2.0609

$*$ m is the delay-partitioning number

However, Theorem 2 successfully enhances the delay bounds compared to the results mentioned in Table I. Also, the results of Theorem 3 clearly provide lager delay bounds than those of Theorem 2, which supports the effectiveness in reducing the conservatism of the stability criterion.

Example 2: Consider the neural networks (6) with the parameters

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$
$$W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \quad K_p = \mathrm{diag}\{0.4,\ 0.8\},$$
$$K_m = \mathrm{diag}\{0,\ 0\}. \tag{69}$$

For this system, by dividing the time-varying delay interval into some subintervals, the maximum delay bounds for guaranteeing the asymptotic stability of the network were presented in [28]. And by dividing delay interval into two and employing different free-weighting matrices at each interval, improved maximum delay bounds were obtained in [22], [23], [25], and [27] when h_D is 0.8, 0.9, and unknown. By application of Theorems 1–3 and Corollaries 1 and 2, our obtained delay bounds and the detailed comparisons with those [25] and [27] are given in Table II. From Table II, Theorem 1 clearly shows less conservatism compared to the results of [22], [23], [25], [27], and [28] in spite of not utilizing the delay-partitioning technique. Furthermore, Theorems 2 and 3 and Corollaries 1 and 2 also verify the effectiveness in improvement of feasible region. In Table III, when $0 \le \dot{h}(t) \le h_D$, another comparison of our results with those of [29] which utilized delay-partitioning approach is shown. Except the results of Theorem 1 when $h_D = 0.8$, all other results obtained by applying the proposed methods give larger delay bounds than those of [29].

Example 3: Consider the neural networks (6) where

$$A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}$$
$$W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}$$
$$W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}$$
$$K_p = \mathrm{diag}\{0.1137,\ 0.1279,\ 0.7994,\ 0.2368\}. \tag{70}$$

In Table IV, when $K_m = \text{diag}\{0,\ 0,\ 0,\ 0\}$ (Case 1), the comparison results on the maximum delay bound allowed via the methods in recent works [20], [23], [24], [26], [28] are presented. From Table IV, it can be seen that Theorem 1 provides larger delay bounds than the existing ones. Also, the obtained results of Theorems 2 and 3 show that the proposed ideas in Theorems 2 and 3 significantly enhance the feasible region of stability criterion compared to those of Theorem 1. And the results of Corollaries 1 and 2 also enhance the feasible region of stability condition compared to those of [23] when $\dot{h}(t) \leq h_D$ and h_D is unknown, respectively. In Table V, when $0 \leq \dot{h}(t) \leq h_D$, another comparison of our results with those of [29] which utilized delay-partitioning approach is shown. For the case when h_D is 0.9 and unknown, all the results obtained by applying the proposed methods give larger delay bounds than those of [29]. When h_D is less than 0.5, the results of Theorems 2 and 3 are larger than those of [29] with the delay-partitioning number 2. When the delay-partitioning number is larger than 2 in [29], the delay bounds of [29] with $h_D = 0$ and $h_D = 0.1$ are larger than our results. It should be noted that, based on the proposed methods, if the delay-partitioning approach is used, then the corresponding delay bounds become large, which will be a future research topic. Lastly, when $K_m = \text{diag}\{-0.4,\ -0.1,\ -0.2,\ -0.3\}$(Case 2), some comparisons of maximum delay bounds are conducted in Table VI, which shows that the proposed methods significantly increase the feasible region of stability compared to those of [20] and [24].

V. Conclusion

In this paper, some delay-dependent stability criteria for neural networks with time-varying delays in which both the upper and lower bounds of delay-derivative were presented. In Theorem 1, by constructing the new augmented LK functional and utilizing some recent results introduced in [33] and [34], the sufficient condition for guaranteeing the asymptotic stability of neural network having time-varying delays in (6) was derived. Based on the results of Theorem 1, by proposing the new idea of dividing the bounding of activation functions into two, an improved stability criterion was proposed in Theorem 2. Also, by constructing new inequalities of activation functions, a further improved stability criterion was presented in Theorem 3. When $\dot{h}(t) \leq h_D$ and h_D are unknown, the corresponding stability conditions were proposed in Corollaries 1 and 2, respectively. Via three numerical examples available in the literature, the improvement of the proposed stability criteria was successfully verified.

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