On the reachable set bounding of uncertain dynamic systems with time-varying delays and disturbances

O.M. Kwon a, S.M. Lee b, Ju H. Park c,⇑

⇑Corresponding author. Tel.: +82 53 810 2491; fax: +82 53 810 4767.
E-mail addresses: madwind@chungbuk.ac.kr (O.M. Kwon), moony@daegu.ac.kr (S.M. Lee), jessie@ynu.ac.kr (J.H. Park).

Article info
Article history:
Received 28 September 2010
Received in revised form 8 March 2011
Accepted 26 April 2011
Available online 4 May 2011

Keywords:
Reachable set
Time-varying delays
Linear matrix inequality
Lyapunov method

1. Introduction

Reachable set bounding was first considered in the late 1960s in the field of state estimation and it has subsequently received a lot of attention with regard to parameter estimation [3]. A reachable set is defined as a set which bounds the state trajectories for systems with disturbances. The ellipsoidal bounding of reachable sets is of practical importance in the design of a controller for dynamic systems with disturbances [6]. For example, as pointed in [6], minimization of an ellipsoidal bound of reachable sets for linear systems with saturating actuators can lead to the design of a controller with a larger gain and this can result in a better performance of the system [7].

On the other hand, it is well-known that occurrence of time-delay may cause the instability or poor performance of dynamic systems and the delay-dependent stability criteria are less conservative than the delay-independent ones. Thus, extensive researches on the delay-dependent stability or stabilization criteria for dynamic systems with time-delay, which give the available information about the maximum allowable time-delay required to guarantee stability, have been conducted by many researchers during the last decade. An overview and recent survey in this field can be obtained from [1,8–11,13,14] and the references therein. Recently, new Lyapunov–Krasovskii's functionals of the form of triple integrals were proposed in [1], and their stability conditions showed an improvement in the maximum delay bound which is an important index for checking of the conservativeness of the delay-dependent stability criteria.

Since an LMI condition for an ellipsoid that bounds the reachable set of linear systems without time-delay was given by Boyd et al. [2], Fridman and Shaked [4] firstly proposed LMI criteria of an ellipsoid that bounds the reachable set of uncertain systems with time-varying delays and bounded peak input based on the Razumikhin theorem. More recently, by the use of...
the Lyapunov–Krasovskii functional method which is commonly used to derive delay-dependent stability criteria, improved conditions were investigated in [6]. However, when the upper bound of the time-derivative delay becomes larger, the results given in [6] are more conservative than the ones presented in [4]. Furthermore, when the information on the time derivative of the time-varying delays is unknown, the methods in [4] cannot be applied. Therefore, there is room for further improvement in the application to dealing with this problem.

Motivated by this situation, improved LMI criteria will be derived in this paper to find an ellipsoidal bound of reachable sets for uncertain systems with the general time-derivative constraints of a time-varying delay has not been considered.

Notation: \( \mathbb{R}^n \) is the n-dimensional Euclidean space, \( \mathbb{R}^{m \times n} \) denotes the set of \( m \times n \) real matrices. \( \| \cdot \| \) refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices \( X \) and \( Y \), the notation \( X \geq Y \) (respectively, \( X \succ Y \)) means that the matrix \( X-Y \) is positive definite, (respectively, nonnegative). \( \text{diag}\{\cdot\} \) denotes the block diagonal matrix. \( \star \) represents the elements below the main diagonal of a symmetric matrix and the superscript ‘\(^T\)’ represents the transpose.

2. Problem statements

Consider the uncertain dynamic systems with time-varying delays and disturbances:

\[
\begin{align*}
\dot{x}(t) &= (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - \hat{h}(t)) + (B + \Delta B(t))w(t), \\
x(s) &= 0, \quad s \in [-h_0, 0],
\end{align*}
\]

(1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( w(t) \in \mathbb{R}^m \) is the disturbance input, \( A, A_d \) and \( B \) are known constant matrices with appropriate dimensions, and \( \Delta A(t), \Delta A_d(t), \) and \( \Delta B(t) \) are uncertainties expressed as a linear convex-hull of matrices \( A_i, A_d_i, B_i \).

Hence

\[
\begin{align*}
\Delta A(t) &= \sum_{i=1}^{N} \rho_i(t)A_i, \\
\Delta A_d(t) &= \sum_{i=1}^{N} \rho_i(t)A_d_i, \\
\Delta B(t) &= \sum_{i=1}^{N} \rho_i(t)B_i,
\end{align*}
\]

(2)

where \( \rho_i(t) \in [0, 1], (i = 1, \ldots, N), \sum_{i=1}^{N} \rho_i(t) = 1, \forall t \geq 0. \)

The disturbances are assumed to be bounded,

\[
w^T(t)w(t) \leq w_m^2, \quad \forall t \geq 0.
\]

(3)

where \( w_m \) is constant.

The delay, \( \hat{h}(t) \), is a time-varying continuous function which satisfies \( 0 \leq \hat{h}(t) \leq h_0 \). Defining \( \hat{h}(t) = \frac{1}{\Delta h}\{ h(t) \} \). In this paper, three cases of time-delay derivative conditions are considered as follows:

- **Case I**: \( -\infty < \hat{h}(t) < \infty \),
- **Case II**: \( -\infty < \hat{h}(t) < h_D \),
- **Case III**: \( h_D < \hat{h}(t) \leq h_{DD} \),

where \( h_D > 0 \), \( h_D \) and \( h_{DD} \) are constants.

Defining an ellipsoid \( \varepsilon \) which bounds the reachable sets of system (1) with the constraints of disturbance and time-varying delays, then

\[
\varepsilon = \{ x \in \mathbb{R}^n : x^T P x \leq 1 \}.
\]

(4)

where \( P > 0 \).

The objective of this paper is to investigate delay-dependent LMI conditions for finding an ellipsoid \( \varepsilon \) (4) that bounds the reachable sets of the system (1).

**Remark 1.** In [4,6], the considered constraints of \( \hat{h}(t) \) were \( -\infty < \hat{h}(t) < \infty \) and \( |\hat{h}(t)| \leq h_D < 1 \), respectively. Therefore, Cases I–III of \( \hat{h}(t) \) are more general than the ones in [4,6].

In order to derive LMI conditions for finding an ellipsoidal bound \( \varepsilon \), the convex-hull representation and properties will be utilized in the main results. For the Case I and II, only \( \hat{h}(t) \) can be represented as convex hull, and for the Case III, whilst \( \hat{h}(t) \) also can be considered as a convex hull.
For the condition \(0 \leq h(t) \leq h_U\) and \(h_{IR} \leq h(t) \leq h_{out}\), \(\nabla_d\) and \(\nabla_h\) are next defined in the set

\[
\Phi_d := \{ \nabla_d | \nabla_d \in \text{Co} \{ \nabla_d^1, \nabla_d^2 \} \},
\]

\[
\Phi_h := \{ \nabla_h | \nabla_h \in \text{Co} \{ \nabla_h^1, \nabla_h^2 \} \},
\]

where \(\text{Co}\) denotes the convex hull, \(\nabla_d^1 = h_{IR}, \nabla_d^2 = h_{out}, \nabla_h^1 = 0,\) and \(\nabla_h^2 = h_U\).

Then, there exist parameters \(\beta_i\) and \(\gamma_i\) where \(\beta_i \geq 0, \gamma_i \geq 0\) for \(i = 1, 2, \sum_{i=1}^2 \beta_i = 1,\) and \(\sum_{i=1}^2 \gamma_i = 1\) such that \(\dot{h}(t)\) and \(h(t)\) can be expressed as a convex combination of the vertex values as follows:

\[
\dot{h}(t) = \sum_{i=1}^2 \beta_i \nabla_d^i, \\
\]

\[
h(t) = \sum_{i=1}^2 \gamma_i \nabla_h^i. \\
\]

If matrices \(M(\dot{h}(t))\) and \(G(h(t))\) are affinely dependent on \(\dot{h}(t)\) and \(h(t)\) respectively, then \(M(\dot{h}(t))\) and \(G(h(t))\) can be expressed as a convex combination of the vertex values, respectively, i.e.,

\[
M(\dot{h}(t)) = \sum_{i=1}^2 \beta_i M(\nabla_d^i), \\
G(h(t)) = \sum_{i=1}^2 \gamma_i G(\nabla_h^i). \\
\]

Before deriving the main results, the following lemmas will be stated.

**Lemma 1** [5]. For any positive-definite matrix \(M \in \mathbb{R}^{n \times n}\), a positive scalar \(\gamma\), and a vector function \(z : [0, \gamma] \rightarrow \mathbb{R}^n\) such that the integrations concerned are well defined, then

\[
\left( \int_0^\gamma z(s)ds \right)^T M \left( \int_0^\gamma z(s)ds \right) \leq \gamma \int_0^\gamma z^T(s)Mz(s)ds.
\]

**Lemma 2** [2]. Let \(V(x(0)) = 0\) and \(w^f(t)w(t) \leq w_m^2\). If

\[
V(x(t)) + xV(x(t)) - \beta w^f(t)w(t) \leq 0, \\
x > 0, \\
\beta > 0,
\]

then,

\[
V(x(t)) \leq \frac{\beta}{x} w_m^2, \quad \forall t \geq 0.
\]

**Lemma 3** [12]. Let \(\zeta \in \mathbb{R}^n, \Phi = \Phi^T \in \mathbb{R}^{n \times n},\) and \(B \in \mathbb{R}^{m \times n}\) such that \(\text{rank}(B) < n\). The following statements are equivalent:

(i) \(\zeta^T \Phi \zeta < 0, \forall B \zeta = 0, \zeta \neq 0,\)

(ii) \((B^+)^T \Phi (B^+) < 0\) where \(B^+\) is a right orthogonal complement of \(B\).

**Lemma 4.** For a positive matrix \(M\), the following inequality holds:

\[
-\frac{(\alpha - \beta)^2}{2} \int_s^x \int_s^x x^T(u)Mx(u)du
ds \leq -\left( \int_s^x \int_s^x x(u)du \right)^T M \left( \int_s^x \int_s^x x(u)du \right).
\]

**Proof.** From Lemma 1, the following inequality holds:

\[
-\left( \int_s^x \int_s^x x^T(u)Mx(u)du \right) \leq -\left( \int_s^x x(u)du \right)^T M \left( \int_s^x x(u)du \right).
\]

By the use of Schur’s Complement [2], inequality (9) is equivalent to

\[
\begin{bmatrix}
- \int_s^x x^T(u)Mx(u)du & (\int_s^x x(u)du)^T \\
- (\alpha - s)M^{-1}
\end{bmatrix} \leq 0.
\]
Integration of the inequality (10) from \( \beta \) to \( \alpha \) yields
\[
\left[ -\int_{\beta}^{\alpha} x^T(u) M x(u) du \left( \int_{\beta}^{\alpha} x(u) du \right)^T - \int_{\beta}^{\alpha} (x - s) M^{-1} ds \right] \leq 0. \tag{11}
\]
From Schur's Complement, inequality (11) is equivalent to inequality (8). This completes the Proof of Lemma 4. \( \square \)

3. Main results

In this section, delay-dependent conditions for finding an ellipsoidal reachable sets that bounds the states of the system (1) will be derived by the Lyapunov stability method. First, for the Case I, the following theorem is considered.

**Theorem 1.** For given \( \alpha > 0 \) and \( h_0 > 0 \), the reachable sets of the system (1) with \( 0 < h(t) \leq h_0 \) and \( -\alpha < h(t) < \alpha \) are bounded by an ellipsoid \( \varepsilon \) defined in (4) if there exist positive matrices \( P, N_1, G = \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \) and any matrices \( L_1, L_2, L_3 \) satisfying the LMIs:

\[
\Sigma^i_1 + \Omega^k < 0 \quad (i = 1, 2, k = 1, \ldots, N),
\]

where

\[
\Sigma^{i(1)}_1 = [\Sigma^{i(1)}_{m,n}], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7.
\]

\[
\Sigma^{i(1)}_1 = x^T + N_1 + h_0^2 G_{11} + e^{-2 h_0} (2 + h_0^2) G_{22}, \quad \Sigma^{i(2)}_1 = -e^{-2 h_0} (2 + h_0^2) G_{22},
\]

\[
\Sigma^{i(3)}_1 = 0, \quad \Sigma^{i(4)}_1 = P + h_0^2 G_{12}, \quad \Sigma^{i(5)}_1 = e^{-2 h_0} (2 + h_0^2) G_{12}, \quad \Sigma^{i(6)}_1 = 0,
\]

\[
\Sigma^{i(7)}_1 = 0, \quad \Sigma^{i(8)}_1 = e^{-2 h_0} (2 + h_0^2) G_{12}, \quad \Sigma^{i(9)}_1 = 0, \quad \Sigma^{i(10)}_1 = h_0^2 G_{22},
\]

\[
\Sigma^{i(11)}_1 = 0, \quad \Sigma^{i(12)}_1 = e^{-2 h_0} (2 + h_0^2) G_{12}, \quad \Sigma^{i(13)}_1 = 0, \quad \Sigma^{i(14)}_1 = h_0^2 G_{22},
\]

\[
\Omega^{k} = [\Omega^{k}_{m,n}], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7.
\]

\[
\Omega^{(1)}_1 = L_1 (A + A_k) + (A + A_k)^T L_1, \quad \Omega^{(2)}_1 = L_1 (A_d + A_{dk}) + (A + A_k)^T L_1,
\]

\[
\Omega^{(3)}_1 = -L_1 (A + A_k)^T L_1, \quad \Omega^{(4)}_1 = -L_1 (A_d + A_{dk}) + (A + A_k)^T L_1,
\]

\[
\Omega^{(5)}_1 = L_2 (B + B_k), \quad \Omega^{(6)}_1 = L_2 (A_{dk} + A_d) + (A + A_k)^T L_2,
\]

\[
\Omega^{(7)}_1 = -L_2 (A_d + A_{dk}) + (A + A_k)^T L_2, \quad \Omega^{(8)}_1 = L_2 (A_{dk} + A_d) + (A + A_k)^T L_2,
\]

\[
\Omega^{(9)}_1 = 0, \quad \Omega^{(10)}_1 = 0.
\]

**Proof.** For positive matrices \( P, N_1, G = \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \), consider the Lyapunov–Krasovskii functional candidate:

\[
V(x(t)) = \sum_{i=1}^{3} V_i(x(t)) \tag{15}
\]

where

\[
V_1(x(t)) = x^T(t) P x(t),
\]

\[
V_2(x(t)) = \int_{t-h_0}^{t} e^{(t-s)} x^T(s) N_1 x(s) ds, \tag{16}
\]

\[
V_3(x(t)) = \int_{t-h_0}^{t} \int_{t-h_0}^{s} e^{(t-u)} x^T(u) G x(u) du ds.
\]

Calculation of the time-derivative of \( V_1(x(t)) \) yields

\[
\dot{V}_1(x(t)) = 2 x^T(t) P x(t). \tag{17}
\]
The time-derivative of $V_2(x(t))$ can now be obtained as

$$V_2(x(t)) = \frac{d}{dt} \left( e^{-at} \int_{t-h}^{t} e^{as} x(s) N_1 x(s) ds \right)$$

$$= -2e^{-at} \int_{t-h}^{t} e^{as} x(s) N_1 x(s) ds + e^{-at} \left[ x^T(t) (e^{at} N_1) x(t) - x^T(t-h) (e^{a(t-h)} N_1) x(t-h) \right]$$

$$= -2V_2(x(t)) + x^T(t) N_1 x(t) - x^T(t-h) e^{-ah} N_1 x(t-h).$$

(18)

Calculation of $V_3(x(t))$ leads to

$$V_3(x(t)) = \frac{d}{dt} \left( hU \int_{t-h}^{t} e^{as} x(s) N_2 x(s) ds \right) = -2V_3(x(t)) + e^{-at} \left[ \int_{t-h}^{t} e^{as} x(s) N_2 x(s) ds \right]$$

$$= -2V_3(x(t)) + e^{-at} \left[ hU \int_{t-h}^{t} x(s) N_2 x(s) ds - hU \int_{t-h}^{t} e^{as} x(s) N_2 x(s) ds \right]$$

$$\leq -2V_3(x(t)) + (hU) x^T(x(t)) N_2 x(x(t)) - e^{-ah} \left[ hU \int_{t-h}^{t} x(s) N_2 x(s) ds \right].$$

(19)

where $-e^{a(t-h)} \leq -e^{-ah}$ was used in (19).

It should be noted that

$$-hU \int_{t-h}^{t} \left[ x(s) \right] ^T N_2 x(s) ds = -hU \int_{t-h}^{t} \left[ x(s) \right] ^T x(s) ds - hU \int_{t-h}^{t} \left[ x(s) \right] ^T hU \int_{t-h}^{t} \left[ x(s) \right] ^T x(s) ds$$

$$\leq -hU x^T(hU) h(t) \left[ x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds$$

$$\leq -hU x^T(hU) h(t) \left[ x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds$$

$$\leq (2 + hU) h(t) \left[ f_{t-h}^{t} x(s) ds \right] ^T x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds$$

(20)

With the similar procedure to that used to obtain (21) and the use of Lemma 1, an upper bound of the second integral term in the right side of Eq. (20) can be estimated to be

$$-hU \int_{t-h}^{t} \left[ x(s) \right] ^T N_2 x(s) ds = -hU x^T(hU) h(t) \left[ x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds$$

$$\leq -(hU - h(t)) \left[ x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds$$

$$\leq (-1 - hU - h(t)) \left[ f_{t-h}^{t} x(s) ds \right] ^T x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds.$$

(21)

With the similar procedure to that used to obtain (21) and the use of Lemma 1, an upper bound of the second integral term in the right side of Eq. (20) can be estimated to be

$$-hU \int_{t-h}^{t} \left[ x(s) \right] ^T N_2 x(s) ds = -(hU - h(t)) \left[ x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds$$

$$\leq -(hU - h(t)) \left[ x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds$$

$$\leq (-1 - hU - h(t)) \left[ f_{t-h}^{t} x(s) ds \right] ^T x(t) - x(t-h) \right] ^T N_2 x(x(t)) ds.$$

(22)

Therefore, from (21) and (22), an upper bound of $V_3(x(t))$ can be further estimated as

$$\hat{V}_3(x(t)) \leq -2V_3(x(t)) + (hU) x^T(x(t)) N_2 x(x(t)) + e^{-ah} \left[ -2 + hU h(t) \right] \left[ f_{t-h}^{t} x(s) ds \right] ^T x(t - x(t-h)) ^T N_2 x(x(t)) ds$$

$$+ (-1 - hU - h(t)) \left[ f_{t-h}^{t} x(s) ds \right] ^T x(t - x(t-h)) - x(t - hU) \right] ^T N_2 x(x(t)) ds.$$

In the upper bound of time-derivative of $V(x(t))$, the following equality is added with free variables $L_i$ ($i = 1, 2, 3$) to be chosen as:

$$0 = 2 \left[ 2x(t)L_1 + x^T(t-h) L_2 + x^T(t)L_3 \right] \times [ \dot{x}(t) + (A + \Delta A) x(t) + (A_d + \Delta A_d) x(t-h) + (B + \Delta B) w(t) ].$$

(23)
Defining an \( z(t) \) as
\[
z^T(t) = \begin{bmatrix} x^T(t) & x^T(t - h(t)) & x^T(t - h_u) & \int_{t-h_u}^t x^T(s)ds & \int_{t-h_0}^t x^T(s)ds & w^T(t) \end{bmatrix},
\]
then, from (15)–(24),
\[
\dot{V}(x(t)) + xV(x(t)) - \frac{\alpha}{W_m} w^T(t)w(t) \leq z^T(t)(\Sigma_1 + \Omega)z(t),
\]
where
\[
\Sigma_1 = \begin{bmatrix} \Sigma_{1[m,n]} \end{bmatrix}, \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7,
\]
\[
\Sigma_{1(1,1)} = \alpha P + N_1 + h_u^2 G_{11} + e^{-zh_u}(2 - h_u^{-1}h(t))G_{22}, \quad \Sigma_{1(1,2)} = -e^{-zh_u}(2 + h_u^{-1}h(t))G_{22},
\]
\[
\Sigma_{1(1,3)} = 0, \quad \Sigma_{1(1,4)} = P + h_u^2 G_{12}, \quad \Sigma_{1(1,5)} = e^{-zh_u}(2 + h_u^{-1}h(t))G_{12}, \quad \Sigma_{1(1,6)} = 0,
\]
\[
\Sigma_{1(1,7)} = 0, \quad \Sigma_{1(1,8)} = e^{-zh_u}(2 + h_u^{-1}h(t))G_{22} + e^{-zh_u}(1 - h_u^{-1}h(t))G_{22},
\]
\[
\Sigma_{1(2,3)} = -e^{-zh_u}(1 - h_u^{-1}h(t))G_{22}, \quad \Sigma_{1(2,4)} = 0, \quad \Sigma_{1(2,5)} = -e^{-zh_u}(2 + h_u^{-1}h(t))G_{12},
\]
\[
\Sigma_{1(2,6)} = e^{-zh_u}(1 - h_u^{-1}h(t))G_{12}, \quad \Sigma_{1(2,7)} = 0, \quad \Sigma_{1(3,3)} = -e^{-zh_u}N_1 + e^{-zh_u}(1 - h_u^{-1}h(t))G_{22},
\]
\[
\Sigma_{1(3,4)} = 0, \quad \Sigma_{1(3,5)} = 0, \quad \Sigma_{1(3,6)} = -e^{-zh_u}(1 - h_u^{-1}h(t))G_{12}, \quad \Sigma_{1(3,7)} = 0, \quad \Sigma_{1(4,4)} = h_u^2 G_{22},
\]
\[
\Sigma_{1(4,5)} = 0, \quad \Sigma_{1(4,6)} = 0, \quad \Sigma_{1(4,7)} = 0, \quad \Sigma_{1(5,5)} = e^{-zh_u}(2 + h_u^{-1}h(t))(e^{zh_u}e^{zh_u})N_2 + G_{11},
\]
\[
\Sigma_{1(5,6)} = 0, \quad \Sigma_{1(5,7)} = 0, \quad \Sigma_{1(6,6)} = e^{-zh_u}(1 - h_u^{-1}h(t))G_{11}, \quad \Sigma_{1(6,7)} = 0, \quad \Sigma_{1(7,7)} = -\frac{\alpha}{W_m} I
\]
and
\[
\Omega = [\Omega_{m,n}], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7,
\]
\[
\Omega_{1,1} = L_1(A + \Delta A(t)) + (A + \Delta A(t))^TL_1, \quad \Omega_{1,2} = L_1(A_d + \Delta A_d(t)) + (A + \Delta A(t))^TL_1,
\]
\[
\Omega_{1,4} = -L_1 + (A + \Delta A(t))^TL_2, \quad \Omega_{1,7} = L_1(B + \Delta B(t)),
\]
\[
\Omega_{2,1} = L_2(A_d + \Delta A_d(t)) + (A_d + \Delta A_d(t))^TL_1, \quad \Omega_{2,2} = -L_2 + (A_d + \Delta A_d(t))^TL_2,
\]
\[
\Omega_{2,7} = L_2(B + \Delta B(t)), \quad \Omega_{4,4} = -L_1 - L_1^T, \quad \Omega_{4,7} = L_3(B + \Delta B(t)),
\]
\[
\Omega_{m,n} = 0 \quad (\text{elsewhere}).
\]
\[\text{It should be noted that} \; \Sigma_1 + \Omega < 0, \text{which means that the inequality} \; \dot{V}(x(t)) + xV(x(t)) - \frac{\alpha}{W_m} w^T(t)w(t) < 0, \text{is affinely dependent on} \; h(t), \Delta A(t), \Delta A_d(t), \text{and} \; \Delta B(t). \text{From the properties of a convex-hull, if the LMI (12) holds, then} \; \Sigma_1 + \Omega < 0 \text{is satisfied. Thus, } \dot{V}(x(t)) \leq \frac{\alpha}{W_m} w^T_m = 1 \text{ by Lemma 2. Also, } V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) \geq V_1(x(t)) = x^T(t)P(x(t)) \text{ since } V_2(x(t)) + V_3(x(t)) \geq 0. \text{Therefore, if the LMI (12) holds, then } x^T(t)P(x(t)) \leq V(x(t)) \leq 1. \text{This completes the proof. } \Box
\]

**Remark 2.** As mentioned in [4,6], a simple approximation to minimize the volume \( \epsilon \) of Eq. (4) in Theorem 1 can be used as follows:

minimize \[\delta,\]
subject to \[
\begin{bmatrix}
\delta I & I \\
I & P
\end{bmatrix} \succeq 0 \quad \text{and LMI (12)}.
\]

Here it should be noted that Theorem 1 does not impose any conditions on the time-derivative of \( h(t) \) in system (1). If \( \dot{h}(t) \) in system (1) satisfies \(-\infty < \dot{h}(t) \leq h_0 \) where \( h_0 > 0 \), then we have the following theorem.

**Theorem 2.** For given \( \alpha > 0, h_u > 0, h_d > 0 \), the reachable sets of the system (1) with \( 0 \leq h(t) \leq h_u \) and \(-\infty < h(t) \leq h_0 \) are bounded by an ellipsoid \( \epsilon \) as defined in (4) providing that there exist positive matrices \( P, N_1, N_2, Q \in R^{[G_{11} \quad G_{12} \quad G_{22}]} \), and any matrices \( L_1, L_2, L_3 \) satisfying the following LMIs:

\[
\Sigma_1^i + \Sigma_2^i + O^k < 0 \quad (i = 1, 2, k = 1, \ldots, N),
\]
\[
\begin{bmatrix}
-xh_u^{-1} e^{zh_u} N_2 + G_{11} & 0 \\
0 & G_{22}
\end{bmatrix} > 0.
\]
where
\[
\Sigma_i^{1} = [\Sigma_{2[m,n]}], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7,
\]
\[
\Sigma_i^{2(1,1)} = N_2, \quad \Sigma_i^{2(2,2)} = -(1 - h_{2})N_2, \quad \Sigma_i^{2(5,5)} = (2 - h_{2}^{-1}b_{1})zh_{2}^{-1}N_2,
\]
\[
\Sigma_i^{2[2m,n]} = 0 \quad \text{(elsewhere)}.
\]
and \( \Sigma_i^{1} \) and \( \Omega^{k} \) have already been defined in (13) and (14), respectively.

**Proof.** For positive matrices \( P, N_1, N_2, \) and \( \mathcal{G} \), consider the Lyapunov–Krasovskii functional candidate:
\[
V(x(t)) = \sum_{i=1}^{4} V_i(x(t)),
\]
where
\[
V_4(x(t)) = \int_{t-h(t)}^{t} x^T(s)N_2x(s)ds
\]
and \( V_i(x(t)) \) \((i = 1, 2, 3)\) are defined in (31).

Calculation of \( V_4(x(t)) \) establishes the relationship
\[
V_4(x(t)) = x^T(t)N_2x(t) - (1 - \dot{h}(t))x^T(t - h(t))N_2x(t - h(t)) \leq x^T(t)N_2x(t) - (1 - h_{2})x^T(t - h(t))N_2x(t - h(t)).
\]
Since the form of \( V_4(x(t)) \) does not contain \(-\alpha V_4(x(t))\) as shown in (33), \( \alpha V_4(x(t)) \) is added to the left side of the condition to produce
\[
\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{W_m} x^T(t)w(t) < 0.
\]
Since the form of \( \int_{t-h(t)}^{t} x^T(s)N_2x(s)ds \) exists \((N > 0)\) exists in the upper bound of \( \dot{V}_3(x(t)) \) so that
\[
\dot{V}_3(x(t)) + \alpha V_3(x(t)) \leq -\alpha V_3(x(t)) + (h_{2}^{2} \left[ x(t) \right]^T \mathcal{G} \left[ x(t) \right] - e^{-\delta h_2} \left( h_{2} \int_{t-h_2}^{t} x(s) \mathcal{G} x(s) ds \right) + \alpha \int_{t-h(t)}^{t} x^T(s)N_2x(s)ds
\]
\[
= -\alpha V_3(x(t)) + (h_{2}^{2} \left[ x(t) \right]^T \mathcal{G} \left[ x(t) \right] - e^{-\delta h_2} \left( h_{2} \int_{t-h_2}^{t} x(s) \mathcal{G} x(s) ds \right)
\]
\[
- e^{-\delta h_2} \left( h_{2} \int_{t-h_2}^{t} x(s) \mathcal{G} x(s) ds \right).
\]
If the inequality (29) in Theorem 2 holds, then, from (21) and (22), Eq. (35) can be further estimated as
\[
\dot{V}_3(x(t)) \leq -\alpha V_3(x(t)) + (h_{2}^{2} \left[ x(t) \right]^T \mathcal{G} \left[ x(t) \right] + e^{-\delta h_2} \left[ -2 + h_{2}^{-1}h(t) \right] \int_{t-h(t)}^{t-h(t)} x(s)ds \mathcal{G} \int_{t-h(t)}^{t-h(t)} x(s)ds
\]
\[
- e^{-\delta h_2} \left( h_{2} \int_{t-h_2}^{t} x(s) \mathcal{G} x(s) ds \right).
\]
From (15)–(24), (31)–(33), and (35), (36), it is found that
\[
\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{W_m} x^T(t)w(t) < \zeta^T(t)(\Sigma_1 + \Sigma_2 + \Omega \zeta(t)).
\]
Here, \( \Sigma_1 \) and \( \Omega \) are as defined in (26) and (27), respectively. Also \( \Sigma_2 \) is given by
\[
\Sigma_2 = [\Sigma_{2[m,n]}], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7,
\]
\[
\Sigma_2^{2(1,1)} = N_2, \quad \Sigma_2^{2(2,2)} = -(1 - h_{2})N_2, \quad \Sigma_2^{2(5,5)} = (2 - h_{2}^{-1}h(t))zh_{2}^{-1}N_2,
\]
\[
\Sigma_2^{2[2m,n]} = 0 \quad \text{(elsewhere)}.
\]
From the properties of the convex-hull, if the LMIs (28) and (29) hold, then \( \Sigma_1 + \Sigma_2 + \Omega < 0 \), which implies that \( \dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{W_m} x^T(t)w(t) < 0 \). Therefore, from Lemma 2, if the LMIs (28) and (29) hold, then \( x^T(t)Px(t) \leq V(x(t)) \leq 1 \) since \( V(x(t)) \geq x^T(t)P\dot{x}(t) \). This completes the proof. \( \square \)
Remark 3. In practical systems, it is more realistic to consider the condition of $\dot{h}(t)$ which has a lower bound and upper bound such as $h_{10} \leq \dot{h}(t) \leq h_{20}$. For this case, if an LMI condition is affinely dependent on $\dot{h}(t)$, then by utilizing convex-hull properties, an improved condition for finding an ellipsoidal reachable set for system (1) can be obtained by considering the equation

$$V_S(x(t)) = \int_{t-h(t)}^{t} x(s)ds$$

which will be introduced in Theorem 3.

Theorem 3. For a given $x > 0$, $h_U > 0$, $h_D$, and $h_{20}$, the reachable sets of the system (1) with $0 < h(t) \leq h_U$ and $h_{20} \leq \dot{h}(t) \leq h_D$, are bounded by an ellipsoid $e$ as defined in (4) if there exist positive matrices $P_i$, $N_i$, $R_i$, $G_i$, $G_{i,2}$, $R = [R_{11} \ R_{12} \ R_{13}]$, and any matrices $L_1$, $L_2$, $L_3$ satisfying the following LMIs:

$$\Sigma_i^j + \Sigma_i^j + \Sigma_i^j + \Omega^k < 0 \quad (i = 1, 2, j = 1, 2, k = 1, \ldots, N),$$

$$-xh_{u}^{-1}e^tN_2 + G_{11} \ G_{22} > 0,$$

where

$$\Sigma_i^j = \left[\Sigma_i^j_{m,n}\right], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7,$$

$$\Sigma_i^j_{1,1} = N_2, \quad \Sigma_i^j_{1,2} = -(1 - \nabla_d)N_2, \quad \Sigma_i^j_{1,3} = (2 - h_{u}^{-1}\nabla_d)xh_{u}^{-1}N_2,$$

$$\Sigma_i^j_{2,0} = 0 \quad (\text{otherwise}),$$

$$\Sigma_i^j_{3,0} = [\Sigma_i^j_{3,0}], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7,$$

$$\Sigma_i^j_{3,3} = \alpha R_{11} + R_{12} + R_{12}^T, \quad \Sigma_i^j_{3,2} = -(1 - \nabla_d)R_{12} + (1 - \nabla_d)R_{13}, \quad \Sigma_i^j_{3,1} = -R_{13},$$

$$\Sigma_i^j_{3,4} = R_{11}, \quad \Sigma_i^j_{3,5} = \alpha R_{12} + R_{22}, \quad \Sigma_i^j_{3,6} = \alpha R_{13} + R_{23}, \quad \Sigma_i^j_{3,7} = 0, \quad \Sigma_i^j_{3,8} = 0, \quad \Sigma_i^j_{3,9} = 0,$$

$$\Sigma_i^j_{3,10} = 0, \quad \Sigma_i^j_{3,11} = -(1 - \nabla_d)R_{22} + (1 - \nabla_d)R_{23}, \quad \Sigma_i^j_{3,12} = -(1 - \nabla_d)R_{23} + (1 - \nabla_d)R_{33},$$

$$\Sigma_i^j_{3,13} = 0, \quad \Sigma_i^j_{3,14} = 0, \quad \Sigma_i^j_{3,15} = -R_{23}, \quad \Sigma_i^j_{3,16} = -R_{33}, \quad \Sigma_i^j_{3,17} = 0, \quad \Sigma_i^j_{3,18} = 0,$$

$$\Sigma_i^j_{3,19} = \alpha R_{33}, \quad \Sigma_i^j_{3,20} = 0, \quad \Sigma_i^j_{3,21} = 0, \quad \Sigma_i^j_{3,22} = 0,$$

and $\Sigma_i$ and $\Omega^k$ are as already defined in (13) and (14), respectively.

Proof. For positive matrices $P_i$, $N_i$, $N_2$, $G_i$, $G_{i,2}$, $R$, $\Sigma_i$, let us take Lyapunov–Krasovskii functional candidate:

$$V(x(t)) = \sum_{i=1}^{\Sigma} V_i(x(t)),$$

where

$$V_S(x(t)) = \int_{t-h(t)}^{t} x(s)ds$$

and $V_i(x(t)) (i = 1, \ldots, 4)$ are as defined in (31). Calculation of $V_S(x(t))$ leads to

$$\dot{V}_S(x(t)) = 2 \left[ \int_{t-h(t)}^{t} x(s)ds \right]^T R \left[ \int_{t-h(t)}^{t} x(s)ds \right]$$

and $V_i(x(t)) (i = 1, \ldots, 4)$ as defined in (31).
From (15)–(24), (31)–(33), (35), (36) and (43)–(45), the inequality
\[ V(x(t)) + zV(x(t)) - \frac{\alpha}{w_n} w^T(t)w(t) < \zeta^T(t)(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Omega)\zeta(t) \]  
(46)
is obtained where \( \Sigma_1 \) is the same one as in (38) except \( \Sigma_{2(2,2)} = -(1 - \dot{h}(t))N_2 \),
\[ \Sigma_3 = [\Sigma_{3(m,n)}], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7, \]
\[ \Sigma_{3(1,1)} = 2R_{11} + R_{12} + R_{12}^T, \quad \Sigma_{3(2,1)} = -(1 - \dot{h}(t))R_{12} + (1 - \dot{h}(t))R_{13}, \quad \Sigma_{3(3,1)} = -R_{13}, \]
\[ \Sigma_{3(1,4)} = R_{11}, \quad \Sigma_{3(2,4)} = 2R_{12} + R_{22}, \quad \Sigma_{3(3,6)} = 2R_{13} + R_{23}, \quad \Sigma_{3(1,7)} = 0, \quad \Sigma_{3(2,2)} = 0, \quad \Sigma_{3(2,3)} = 0, \]
\[ \Sigma_{3(2,4)} = 0, \quad \Sigma_{3(2,5)} = -(1 - \dot{h}(t))R_{22} + (1 - \dot{h}(t))R_{23}^T, \quad \Sigma_{3(2,6)} = -(1 - \dot{h}(t))R_{23} + (1 - \dot{h}(t))R_{33}, \]
\[ \Sigma_{3(2,7)} = 0, \quad \Sigma_{3(3,3)} = 0, \quad \Sigma_{3(3,4)} = 0, \quad \Sigma_{3(3,5)} = -R_{33}^T, \quad \Sigma_{3(3,6)} = -R_{33}, \quad \Sigma_{3(3,7)} = 0, \quad \Sigma_{3(4,4)} = 0, \]
\[ \Sigma_{3(4,5)} = R_{12}, \quad \Sigma_{3(4,6)} = R_{13}, \quad \Sigma_{3(4,7)} = 0, \quad \Sigma_{3(5,5)} = 2R_{22}, \quad \Sigma_{3(5,6)} = 2R_{23}, \quad \Sigma_{3(5,7)} = 0, \]
\[ \Sigma_{3(6,6)} = 2R_{33}, \quad \Sigma_{3(6,7)} = 0, \quad \Sigma_{3(7,7)} = 0, \]
and \( \Sigma_1 \) and \( \Omega \) are as defined in (13) and (14), respectively. By applying a procedure similar to the proof of Theorem 2, if the LMIs (40) and (41) hold, then \( x^T(t)P(x(t)) \leq V(x(t)) \leq 1 \). This completes the proof. \( \Box \)

Remark 4. Very recently, an improved condition for finding an ellipsoidal volume (4) for system (1) with the condition \( 0 \leq h(t) \leq h_0 \) and \( |\dot{h}(t)| \leq h_0 \) was proposed in [6]. Here it is noted that the considered condition for time-derivative delay in Theorem 3 is \( h_0 \leq |\dot{h}(t)| \leq h_0 \), which is a more general case than \( |\dot{h}(t)| \leq h_0 \). In [6], the proposed Lyapunov–Krasovskii’s functional only has the form of \( \int_{-h(t)}^{t} \int_{-h(t)}^{t} x^T(u)G_{11}X(u)duds \) and the term \( \int_{-h(t)}^{t} \int_{-h(t)}^{t} x^T(u)G_{11}X(u)duds \) was not considered. It is also noted that the \( V_6(x(t)) \) term in (15) contains the form of \( \int_{-h(t)}^{t} \int_{-h(t)}^{t} x^T(u)G_{11}X(u)duds \). Therefore, the proposed \( V_6(x(t)) \) utilizes fully past information on \( x(t) \) for \( 0 \leq h(t) \leq h_0 \). Also, in obtained time-derivative of \( V(x(t)) \) of [6], the term including \( h(t) \) was estimated using the well-known fact \( 2a'b \leq a^Tb^\top b \leq b^Tb \). However, Theorem 3 does not use this fact to estimate the terms which include \( h(t) \). These two differences cause the proposed Theorem 3 to improve the feasible region of an ellipsoidal reachable set for the same system. In Section 4, the results obtained will be compared with the results of [6].

Remark 5. In order to improve the feasible region of the stability condition of the time-delay systems, the form of the triple integral Lyapunov–Krasovskii’s functional was proposed in [1]. An important question is that the form of triple integral Lyapunov–Krasovskii’s functional can be applied to find an ellipsoidal volume (4) for system (1) with the condition \( 0 \leq h(t) \leq h_0 \) and \( |\dot{h}(t)| \leq h_0 \). Motivated by this aim, in Theorem 4, by consideration of the following Lyapunov–Krasovskii’s functional
\[ V_6(x(t)) = (h_0^2/2) \int_{-h_0}^{h_0} \int_{-h_0}^{h_0} \int_{-h_0}^{h_0} e^{x^T(-t)(\nu)}G_{11}x(\nu)d\nu d\nu d\nu, \]  
(48)
a further improved condition will be introduced.

Theorem 4. For given \( \alpha > 0 \), \( h_0 > 0 \), \( h_{2\delta} \) and \( h_{2\delta} \), the reachable sets of the system (1) with \( 0 \leq h(t) \leq h_0 \) and \( |\dot{h}(t)| \leq h_0 \) are bounded by an ellipsoid \( \varepsilon \) defined in (4) provided that there exist positive matrices \( P, N_1, N_2, \ G = \left[ \begin{array}{cc} G_{11} & G_{12} \\ \ast & G_{22} \end{array} \right] \),
\[ R = \left[ \begin{array}{ccc} R_{11} & R_{12} & R_{13} \\ R_{22} & R_{23} & R_{23} \\ \ast & \ast & R_{33} \end{array} \right], \ G_b \) and any matrices \( L_1, L_2, L_3 \) satisfying the following LMIs:
\[ \Sigma_i^j + \Sigma_2^j + \Sigma_3^j + \Sigma_4^i + \Omega^k < 0 \quad (i = 1, 2, j = 1, 2, k = 1, \ldots, N), \]
\[ \left[ -2h_0^{-1}e^{-\lambda h_0}N_2 + G_{11} \quad G_{12} \\ \ast & G_{22} \right] > 0, \]  
(50)
where
\[ \Sigma_4 = [\Sigma_{4(m,n)}], \quad m = 1, \ldots, 7, \quad n = 1, \ldots, 7, \]
\[ \Sigma_{4(1,1)} = -h_0^2e^{-\lambda h_0}G_{11}, \quad \Sigma_{4(1,5)} = h_0e^{-\lambda h_0}G_{13}, \quad \Sigma_{4(1,6)} = h_0e^{-\lambda h_0}G_{12}, \quad \Sigma_{4(4,4)} = (h_0/4)^2G_{13}, \]
\[ \Sigma_{4(5,5)} = -e^{-\lambda h_0}G_{12}, \quad \Sigma_{4(5,6)} = -e^{-\lambda h_0}G_{13}, \quad \Sigma_{4(6,6)} = -e^{-\lambda h_0}G_{12}, \]
(51)
\[ \Sigma_{4(m,n)} = 0 \quad \text{(elsewhere)} \]
and other notations are as defined in Theorem 3.
Proof. For positive matrices \(P, N_1, N_2, G, R\), and \(G_3\), taking the Lyapunov–Krasovskii functional candidate:

\[
V(x(t)) = \sum_{i=1}^6 V_i(x(t)),
\]

where

\[
V_6(x(t)) = \left(\frac{h_0^2}{2}\right) \int_{t-h_0}^t \int_u^t e^{(t-u)\xi(t)v} G_3 \xi(t) v \, d\xi(t) v \, du \, ds
\]

and \(V_i(x(t)) \ (i = 1, \ldots, 5)\) are the same ones as in (43).

By calculation of \(V_6(x(t))\), it is found that

\[
\dot{V}_6(x(t)) = -2V_6(x(t)) + \frac{e^{-2t}}{dt} \left\{ \frac{h_0^2}{2} \int_{t-h_0}^t \int_u^t e^{(t-u)\xi(t)v} G_3 \xi(t) v \, d\xi(t) v \, du \, ds \right\}
\]

\[
= -2V_6(x(t)) + e^{-2t} \left\{ \frac{h_0^2}{4} \xi(t) \xi(t) \gamma(t) - \frac{(h_0^2/2) e^{-2\xi(t)}}{h_0 t} \int_{t-h_0}^t \int_u^t e^{(t-u)\xi(t)v} G_3 \xi(t) v \, d\xi(t) v \, du \, ds \right\}
\]

\[
\leq -2V_6(x(t)) + \frac{h_0^2}{4} \xi(t) \xi(t) \gamma(t) - \frac{(h_0^2/2) e^{-2\xi(t)}}{h_0 t} \int_{t-h_0}^t \xi(t) \gamma(t) \, du \, ds \int_{t-h_0}^t \xi(t) \gamma(t) \, du \, ds \int_{t-h_0}^t \xi(t) \gamma(t) \, du \, ds
\]

\[
= -2V_6(x(t)) + \frac{h_0^2}{4} \xi(t) \xi(t) \gamma(t) - \frac{(h_0^2/2) e^{-2\xi(t)}}{h_0 t} \int_{t-h_0}^t \xi(t) \gamma(t) \, du \, ds \int_{t-h_0}^t \xi(t) \gamma(t) \, du \, ds
\]

\[
\times \left( h_0 \xi(t) - \int_{t-h_0}^t \xi(t) \gamma(t) \, du \, ds \right)
\]

where Lemma 4 was used in (54).

By use of the procedure in the proof of Theorem 2, if the LMIs (49) and (50) hold, then \(\xi^T(t) P \xi(t) \leq V(x(t)) \leq 1\). This completes the proof. \(\square\)

Remark 6. In Theorems 1–4, free variables \(L_i \ (i = 1, 2, 3)\) are used. However, by the use of Lemma 3, this can be eliminated as Corollary 1. For simplicity, only Theorem 1 without \(L_i \ (i = 1, 2, 3)\) will be shown in Corollary 1.

Corollary 1. For given \(\alpha > 0\) and \(h_0 > 0\), the reachable sets of the system (1) with \(0 \leq h(t) \leq h_0\) and \(-\infty < h(t) < \infty\) are bounded by an ellipsoid \(e^T e\) defined in (4) if there exist positive matrices \(P, N_1, \) and \(G = \begin{bmatrix} G_{11} & G_{12} \\ * & G_{22} \end{bmatrix}\) satisfying the following LMIs:

\[
(N_i^T)\Sigma_i \Sigma_i^T (N_i^T) < 0 \quad (i = 1, 2), \quad k = 1, \ldots, N)
\]

where \(N_i^T\) is the orthogonal complement of \(I_k\),

\[
I_k = [A + A_k \quad A_d + A_{dk} \quad 0 \quad -I \quad 0 \quad 0 \quad B + B_k]
\]

and \(\Sigma_i^T\) has been defined in Theorem 1.

Proof. Instead of (23), consider the zero equation

\[
0 = \Gamma \zeta(t),
\]

where

\[
\Gamma = [A + \Delta A(t) \quad A_d + \Delta A_d(t) \quad 0 \quad -I \quad 0 \quad 0 \quad B + \Delta B(t)].
\]

By utilization of Lemma 3 and convex-hull properties, the proof of Corollary 1 follows straightforwardly from the proof of Theorem 1. So, the proof is not repeated here. \(\square\)

Remark 7. In [4,6], the tuning parameters are five and one, respectively. The proposed Theorems 1–4 has only the one tuning parameter \(\alpha\). Therefore, as mentioned in [6], a feasibility check of Theorems 1–4 can be numerically tractable.
4. Numerical examples

Example 1. Consider the uncertain time-varying delayed system (1) utilized in [4,6] with the parameters

\[
A = \begin{bmatrix}
-2 & 0 \\
0 & -0.9 + \rho
\end{bmatrix}, \quad A_d = \begin{bmatrix}
-1 & 0 \\
-1 & -1 + 0.5\rho
\end{bmatrix}, \quad B = \begin{bmatrix}
-0.5 \\
1
\end{bmatrix}.
\]

(59)

where \(0 < h(t) < h_0\), \(|\dot{h}(t)| < h_0\), \(|\rho| < 0.2\) and \(w^T(t)w(t) \leq w_0^2 = 1\).

Then, matrices \(A_i\), \(A_{di}\) and \(B_i\) can be obtained in the form

\[
A_1 = \begin{bmatrix}
0 & 0 \\
0 & 0.2
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 0 \\
0 & -0.2
\end{bmatrix}, \quad A_{d1} = \begin{bmatrix}
0 & 0 \\
0 & 0.1
\end{bmatrix}, \quad A_{d2} = \begin{bmatrix}
0 & 0 \\
0 & -0.1
\end{bmatrix}, \quad B_i(i = 1, 2) = 0.
\]

(60)

From Remark 1, by applying Theorems 1–4 to the system (59) when \(h_0 = 0.7\) and \(h_0 = 0.75\), the sizes (\(\beta\)) of the ellipsoidal bound of the reachable sets obtained for different \(h_0\) are listed as Tables 1 and 2, respectively. From Tables 1 and 2, even if the most restrictive condition \(-\infty < h(t) < \infty\), which means that the time-derivative information of \(h(t)\) is unknown, is considered in Theorem 1, the results obtained are less conservative than those of [4,6]. Also, by applying Corollary 1 to the system (59), it can be seen that the sizes \(\beta\) are the same ones as in Theorem 1 in Tables 1 and 2. It should be noted that the number of decision variables of Corollary 1 was 17, whilst the number of variable in [6] was 23. This situation meant that Corollary 1 with fewer decision variables than those of [6] gives larger feasible regions than are found in the results presented in [6] in spite of the most restrictive constraints of \(h(t)\) being taken into consideration.

When \(-\infty < h(t) \leq h_0\) is considered in Theorem 2, improved results are obtained as shown in Tables 1 and 2. When \(\dot{h}(t) \leq h_0\) is considered in Theorem 3, it can be seen that less conservative results than those of Theorems 1 and 2 are obtained in Theorem 3. Lastly, by considering a triple integral form of the Lyapunov–Krasovskii’s functionals (53) when \(h(t) \leq h_0\), slightly improved results can be obtained.

Example 2. Consider the practical system which is the satellite system [15] shown in Fig. 1. The satellite system has two rigid bodies joined by a flexible link. The dynamic equations of this system are

### Table 1

The sizes (\(\beta\)) of ellipsoidal bound with \(h_0 = 0.7\) and different \(h_0\) (Example 1).

| \(h_0\) | Kim [6] \((|h(t)| \leq h_0)\) | Fridman [4] \((\text{unknown})\) | Theorem 1 \((\text{unknown})\) | Theorem 2 \((-\infty < h(t) \leq h_0)\) | Theorem 3 \((|h(t)| \leq h_0)\) | Theorem 4 \((|h(t)| \leq h_0)\) |
|---|---|---|---|---|---|---|
| 0 | 2.97 | – | – | 1.9151 | 1.7142 | 1.7103 |
| 0.1 | 3.00 | – | – | 1.9475 | 1.7795 | 1.7728 |
| 0.2 | 3.85 | – | – | 1.9818 | 1.8088 | 1.8032 |
| 0.3 | 4.85 | – | – | 2.0182 | 1.8251 | 1.8199 |
| 0.4 | 6.93 | – | – | 2.0558 | 1.8349 | 1.8344 |
| 0.5 | 12.84 | – | – | 2.0901 | 1.8535 | 1.8498 |
| 0.6 | 53.86 | – | – | 2.1123 | 1.8703 | 1.8674 |
| 0.7 | – | – | – | 2.1139 | 1.8901 | 1.8881 |
| 0.8 | – | – | – | 2.1139 | 1.9142 | 1.9131 |
| 0.9 | – | – | – | 2.1139 | 1.9438 | 1.9433 |
| Unknown | – | 19.71 | 2.1139 | – | – | – |

### Table 2

The sizes (\(\beta\)) of ellipsoidal bound with \(h_0 = 0.75\) and different \(h_0\) (Example 1).

| \(h_0\) | Kim [6] \((|h(t)| \leq h_0)\) | Fridman [4] \((\text{unknown})\) | Theorem 1 \((\text{unknown})\) | Theorem 2 \((-\infty < h(t) \leq h_0)\) | Theorem 3 \((|h(t)| \leq h_0)\) | Theorem 4 \((|h(t)| \leq h_0)\) |
|---|---|---|---|---|---|---|
| 0 | 3.34 | – | – | 2.3199 | 2.0137 | 2.0100 |
| 0.1 | 3.79 | – | – | 2.3762 | 2.0953 | 2.0866 |
| 0.2 | 4.53 | – | – | 2.4287 | 2.1306 | 2.1229 |
| 0.3 | 5.88 | – | – | 2.4845 | 2.1541 | 2.1464 |
| 0.4 | 8.85 | – | – | 2.5441 | 2.1770 | 2.1692 |
| 0.5 | 18.36 | – | – | 2.6035 | 2.2022 | 2.1947 |
| 0.6 | 127.70 | – | – | 2.6520 | 2.2318 | 2.2250 |
| 0.7 | – | – | – | 2.6654 | 2.2674 | 2.2625 |
| 0.8 | – | – | – | 2.6654 | 2.3130 | 2.3101 |
| 0.9 | – | – | – | 2.6654 | 2.3738 | 2.3725 |
| Unknown | – | 65.42 | 2.6654 | – | – | – |
$J_1 \ddot{\theta}_1(t) + f(\dot{\theta}_1(t) - \dot{\theta}_2(t)) + k(\theta_1(t) - \theta_2(t)) = u(t) + w(t)$,

$J_2 \ddot{\theta}_2(t) + f(\dot{\theta}_1(t) - \dot{\theta}_2(t)) + k(\theta_1(t) - \theta_2(t)) = 0. \quad (61)$

where $J_i (i = 1, 2)$ are the moments of inertia of the two bodies (the main body and the instrumentation module), $f$ is a viscous damping, $k$ is a torque constant, $\theta_i(t) (i = 1, 2)$ are the yaw angles for the two bodies, $u(t)$ is a control input and $w(t)$ is a disturbance. Assume $J_i (i = 1, 2) = 1$, $k = 0.09$, $f = 0.004$, and state vector $x(t) = [x_1(t) x_2(t) x_3(t) x_4(t)]^T = [\theta_1(t) \dot{\theta}_1(t) \theta_2(t) \dot{\theta}_2(t)]^T$. Let us choose the control law as $u(t) = Kx(t - h(t))$ where $K = [-3.3092 -0.7443 -2.5909 -8.0395]$ used in [15]. Then, system (61) can be represented by

$$\dot{x}(t) = Ax(t) + A_\delta x(t - h(t)) + Bw(t),$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0900 & 0.0900 & -0.0040 & 0.0040 \\ 0.0900 & -0.0900 & 0.0040 & -0.0040 \end{bmatrix}, \quad A_\delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3.3092 & -0.7443 & -2.5909 & -8.0395 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$
Assuming that $h_D$ is unknown. Application of Theorem 1 to system (61) when $h_U = 0.3$ yields $\delta = 17.9171$ and the corresponding matrix $P$ is

$$P = \begin{bmatrix} 0.4367 & 0.1507 & 0.1046 & 1.0434 \\ 0.1507 & 0.3075 & 0.0214 & 0.5640 \\ 0.1046 & 0.0214 & 0.0867 & 0.2812 \\ 1.0434 & 0.5640 & 0.2812 & 4.5204 \end{bmatrix}.$$
In order to check the validity of Theorem 1, it is assumed that the following $w(t)$
\[
    w(t) = \begin{cases} 
        1, & 1 \leq t \leq 6, \\
        0, & \text{otherwise} 
    \end{cases}
\]
is applied to the system (61). In the numerical simulations, it is assumed further that $h(t)$ is $0.3 \sin^2(100t)$ and that fourth-order Runge–Kutta method is applied to solve the systems with a time step size of 0.0001. Figs. 2 and 3 show the state response of the system that was obtained (61) and the plot of $x'(t)Px(t)$, respectively. From Fig. 3, it can be seen that $x'(t)Px(t)$ is always less than one, which verifies the validity of the proposed Theorem 1.

5. Conclusion

In this paper, improved LMI conditions for finding an ellipsoidal bound of the reachable set of uncertain systems with time-varying delays and disturbances have been proposed. The constraints of $h(t)$ that were considered are more general ones than the conditions published in the existing literature up to now. In order to show the improved feasible regions arising with the proposed methods, the numerical example used in [4,6] was considered and showed an improvement of the presented LMI conditions even when information on $h(t)$ is unknown. Also the satellite system was considered to show the effectiveness of the proposed methods.

Acknowledgement

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0009273).

References