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# Stability and stabilization for discrete-time systems with time-varying delays via augmented Lyapunov–Krasovskii functional

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**Abstract**

This paper is concerned with the stability and stabilization problems for discrete-time systems with interval time-varying delays. By construction of an augmented Lyapunov–Krasovskii functional and utilization of zero equalities, improved delay-dependent criteria for asymptotic stability of the systems are derived in terms of linear matrix inequalities (LMIs). Based on the proposed stability criteria, a sufficient condition for designing feedback gains of time-delayed controllers which guarantee the stability of the concerned system is presented. Through three numerical examples, the effectiveness to enhance the feasible region of the proposed criteria is demonstrated.


## 1. Introduction

Since most real systems use digital computers (usually microprocessors or microcontrollers) with the necessary input/output hardware to implement the systems, the stability issue of discrete-time systems has been one of the fundamental research fields in the control community

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because of their diverse applications in engineering fields [1]. On the other hand, time-delays often occur in many physical systems such as electric power systems with lossless transmission lines, transport and communication systems, tele-manipulation systems, and so on [2–7]. It is well known that the presence of time-delays in a system causes undesirable dynamic behaviors such as performance degradation and instability of the systems. Therefore, stability and stabilization problems of the discrete-time systems with time-varying delays have been put time and efforts into by many researchers [8–19] because stability analysis of the concerned system is an essential requirement in designing a control system.

The main concern in the stability or stabilization of discrete-time systems with time-varying delays is to enlarge the feasible region which guarantees the asymptotic stability of the concerned system. One of the important index for checking the enhancement of the feasible region is to get maximum delay bounds. In this regard, various approaches to stability analysis for discrete-time systems with time-delays have been investigated in the literature [8–14]. In [8], the stability criterion which is dependent on minimum and maximum delay bounds was presented by use of an inequality developed by Moon et al. [20], and stabilization conditions by static and dynamic output-feedback controllers were further derived in the framework of LMIs. Based on augmented Lyapunov–Krasovskii functional, Gao and Chen [9] proposed some new conditions for guaranteeing the asymptotic stability of discrete-time systems with time-varying delays by circumventing the utilization of some bounding inequalities for cross products between the two vectors and considering some ignored terms. In [10], an improved delay-dependent stability criterion was presented and a sufficient condition for the solvability of the stabilization for discrete-time linear systems via time-delayed controllers was proposed. By utilizing the S-procedure and some inequality techniques, a novel stability condition without considering slack variables, which reduces the computational burden of numerical computation, was derived in [11]. In [12], a delay-partitioning approach, which divides time-delays interval into some subintervals, was applied to obtain stability criteria for discrete-time systems for the first time. Recently, a reciprocally convex approach [21] was utilized to reduce the conservatism of the stability criterion in [13]. Very recently, by using minimal number of slack matrix variables, a less conservative stability criterion was derived in [14]. However, there are rooms for further improvements to reduce the conservatism of stability criteria.

On the other hand, as pointed in [10] and [15], there is often a system that the current system state is not delayed in time but time-delays may exist in a channel from the system to controller. A typical example of such dynamical systems with a time-varying communication delay is a networked control system [16]. Thus, in this case, a time-delayed controller can be naturally considered and it is worth to investigate a design problem of a time-delayed controller.

Motivated by this discussed above, the problems of stability and stabilization with a time-delayed controller will be considered. First, by construction of a new augmented Lyapunov–Krasovskii functional and utilization of reciprocally convex approach introduced by [21], a new stability criterion is derived in Theorem 1. It should be pointed out that an upper bound technique for double summation terms in stability analysis of discrete-time systems with time-delays has not been fully investigated while a bounding method for double integral terms in stability analysis of continuous-time systems with time-delays were developed and well known (for example, see [22] and [23]). In order to solve this problem, a new version of Jenson inequality which estimates double summation terms will be introduced in Lemma 1 and then utilized it to obtain a new stability condition. Based on the results of Theorem 1 and motivated by the results [18], a further improved stability criterion will be proposed in Theorem 2 by applying zero equalities to the results of Theorem 1. Finally, a stabilization criterion for designing a gain of time-delayed controller will be presented in Theorem 3. Through three numerical examples, the effectiveness and advantages of the proposed stability and stabilization criteria will be shown by comparing the maximum delay bounds which guarantees the asymptotic stability.

*Notation*: The notations used throughout this paper are fairly standard. $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ denotes the sets of $m\times n$ real matrices. The subscript "T" stands for matrix transpose. For symmetric matrices $X$ and $Y$, $X>Y$ (respectively, $X\geq Y$) means that the matrix $X-Y$ is positive definite (respectively, non-negative). $X^{\perp}$ denotes a basis for the null-space of $X$. $I_n$, $0_n$ and $0_{m\times n}$ denote $n\times n$ identity matrix, $n\times n$ and $m\times n$ zero matrices, respectively. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix norm. $diag\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix.

## 2. Problem statements

Consider the following discrete-time systems with interval time-varying delays

$$\begin{aligned} &x(k+1)=Ax(k)+Bu(k),\\ &u(k)=Kx(k-h(k)),\\ &x(k)=\phi(k), k=-h_M,-h_M+1,\ldots,0, \end{aligned} \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state vector, $u(k)\in\mathbb{R}^m$ is the control input vector, $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$ are known constant matrices, $K$ is the controller gain which will be determined, $\phi(k)$ is the initial function of system (1), and the delay $h(k)$ is interval time-varying delays satisfying

$$0<h_m\leq h(k)\leq h_M,$$

where $h_m$ and $h_M$ are non-negative integer numbers. The aim of this paper is to develop delay-dependent stability analysis and control synthesis of the system (1). In order to do this, the following lemmas are needed.

**Lemma 1.** *For $0<M=M^T\in\mathbb{R}^{n\times n}$, two integers $h_m$ and $h_M$ satisfying $h_m<h_M$, and a vector function $x(s):[k-h_M,k-h_m-1]\to\mathbb{R}^n$, the following inequalities hold*:

$$-(h_M-h_m)\sum_{s=k-h_M}^{k-h_m-1}x^T(s)Mx(s)\leq-\left(\sum_{s=k-h_M}^{k-h_m-1}x(s)\right)^T M\left(\sum_{s=k-h_M}^{k-h_m-1}x(s),\right), \tag{2}$$

$$\begin{aligned} &-\left(\frac{(h_M-h_m)(h_M-h_m+1)}{2}\right)\sum_{s=-h_M}^{-h_m-1}\sum_{u=k+s}^{k-h_m-1}x^T(u)Mx(u)\\ &\quad\leq-\left(\sum_{s=-h_M}^{-h_m-1}\sum_{u=k+s}^{k-h_m-1}x(u)\right)^T M\left(\sum_{s=-h_M}^{-h_m-1}\sum_{u=k+s}^{k-h_m-1}x(u)\right). \end{aligned} \tag{3}$$

**Proof.** According to the work in [19] and by Schur complement, it can be obtained that

$$\begin{bmatrix} x^T(s)Mx(s) & x^T(s)\\ \star & M^{-1}\end{bmatrix}\geq 0, \tag{4}$$

for any $k-h_M \le s \le k-h_m-1$. Summing the above inequality from $k-h_M$ to $k-h_m-1$ leads to

$$\begin{bmatrix} \sum_{s=k-h_M}^{k-h_m-1} x^T(s)Mx(s) & \sum_{s=k-h_M}^{k-h_m-1} x^T(s) \\ \star & (h_M-h_m)M^{-1} \end{bmatrix} \ge 0. \tag{5}$$

Once again, by Schur complement, it follows that

$$\sum_{s=k-h_M}^{k-h_m-1} x^T(s)Mx(s) \ge \left(\sum_{s=k-h_M}^{k-h_m-1} x(s)\right)^T ((h_M-h_m)^{-1}M)\left(\sum_{s=k-h_M}^{k-h_m-1} x(s)\right), \tag{6}$$

which is equivalent to the inequality (2). Next, we will proof the inequality (3). With $k+s \le u \le k-h_m-1$ and $-h_M \le s \le -h_m-1$, by changing the variable $s$ into $u$ in Eq. (4), it can be obtained that

$$\begin{bmatrix} x^T(u)Mx(u) & x^T(u) \\ \star & M^{-1} \end{bmatrix} \ge 0. \tag{7}$$

By summing the above inequality from $k+s$ to $k-h_m-1$, we have

$$\begin{bmatrix} \sum_{u=k+s}^{k-h_m-1} x^T(u)Mx(u) & \sum_{u=k+s}^{k-h_m-1} x^T(u) \\ \star & (-s-h_m)M^{-1} \end{bmatrix} \ge 0. \tag{8}$$

Finally, by summing the above inequality from $-h_M$ to $-h_m-1$, the following inequality holds:

$$\begin{bmatrix} \sum_{s=-h_M}^{-h_m-1} \sum_{u=k+s}^{k-h_m-1} x^T(u)Mx(u) & \sum_{s=-h_M}^{-h_m-1} \sum_{u=k+s}^{k-h_m-1} x^T(u) \\ \star & \sum_{s=-h_M}^{-h_m-1} (-s-h_m)M^{-1} \end{bmatrix} \ge 0, \tag{9}$$

which is equivalent to the inequality (3) by Schur complement. This completes the proof.

**Lemma 2** (*Finsler's lemma [24]*). *Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $\Upsilon \in \mathbb{R}^{m\times n}$ such that rank$(\Upsilon) < n$. The following statements are equivalent*:

(i) $\zeta^T \Phi \zeta < 0,\ \forall \Upsilon\zeta = 0,\ \zeta \ne 0$,
(ii) ${\Upsilon^{\perp}}^T \Phi \Upsilon^{\perp} < 0$,
(iii) $\exists \mathcal{X} \in \mathbb{R}^{n\times m} : \Phi + \mathcal{X}\Upsilon + \Upsilon^T \mathcal{X}^T < 0$.

## 3. Main results

In this section, new stability and stabilization criteria for system (1) will be derived by use of Lyapunov method and LMI framework [25]. For the sake of simplicity on matrix representation, $e_i \in \mathbb{R}^{(10n)\times n} (i = 1, \ldots, 10)$, e.g., $e_2 = [0_n, I_n, 0_n, \ldots, 0_n]^T$ are defined as block entry matrices. The notations of several matrices are defined as

$$\Delta x(k) = x(k+1) - x(k),$$

$$\zeta(k) = \left[x^T(k), x^T(k-h_m), x^T(k-h(k)), x^T(k-h_M), \Delta x^T(k), \Delta x^T(k-h_m),\right.$$

$$\left.\Delta x^T(k-h_M), \sum_{s=k-h_m}^{k-1} x^T(s), \sum_{s=k-h(k)}^{k-h_m-1} x^T(s), \sum_{s=k-h_M}^{k-h(k)-1} x^T(s)\right]^T,$$

$$\Upsilon = [(A-I_n), 0_n, BK, 0_n, -I_n, 0_n, 0_n, 0_n, 0_n, 0_n],$$
$$\alpha(k) = \left[x^T(k), x^T(k-h_m), x^T(k-h_M), \sum_{s=k-h_m}^{k-1} x^T(s), \sum_{s=k-h_M}^{k-h_m-1} x^T(s)\right]^T,$$
$$\beta(k) = [x^T(k), \Delta x^T(k)],$$
$$\Pi_1 = [e_1 + e_5, e_2 + e_6, e_4 + e_7, e_1 - e_2 + e_8, e_2 - e_4 + e_9 + e_{10}],$$
$$\Pi_2 = [e_1, e_2, e_4, e_8, e_9 + e_{10}], \quad \Pi_3 = [e_1, e_5], \quad \Pi_4 = [e_2, e_6], \quad \Pi_5 = [e_4, e_7],$$
$$\Pi_6 = [e_8, e_1 - e_2], \quad \Pi_7 = [e_9, e_2 - e_3, e_{10}, e_3 - e_4],$$
$$\Xi_1 = \Pi_1 \mathcal{R} \Pi_1^T - \Pi_2 \mathcal{R} \Pi_2^T, \Xi_2 = \Pi_3 \mathcal{N} \Pi_3^T + \Pi_4(-\mathcal{N} + \mathcal{M})\Pi_4^T - \Pi_5 \mathcal{M} \Pi_5^T,$$
$$\Xi_3 = (h_m^2)\Pi_3 \mathcal{Q}_1 \Pi_3^T + (h_M - h_m)^2 \Pi_4 \mathcal{Q}_2 \Pi_4^T,$$
$$\Xi_4 = \left(\frac{h_m(h_m+1)}{2}\right)^2 e_5 Q_3 e_5^T + \left(\left(\frac{h_M - h_m}{2}\right)(h_M - h_m + 1)\right)^2 e_6 Q_3 e_6^T,$$
$$-(h_m e_1 - e_8) Q_3 (h_m e_1 - e_8)^T - ((h_M - h_m)e_2 - e_9 - e_{10}) Q_4 ((h_M - h_m)e_2 - e_9 - e_{10})^T,$$
$$\Psi = -\Pi_6 \mathcal{Q}_1 \Pi_6^T - \Pi_7 \begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix} \Pi_7^T,$$
$$\Phi = \sum_{i=1}^{4} \Xi_i. \tag{10}$$

Now, we have the following theorem.

**Theorem 1.** *For given integers $h_m$, $h_M$ satisfying $0 < h_m < h_M$ and a feedback gain matrix K, the system (1) is asymptotically stable for $h_m \le h(k) \le h_M$, if there exist positive definite matrices $\mathcal{R} \in \mathbb{R}^{5n \times 5n}$, $\mathcal{N} \in \mathbb{R}^{2n \times 2n}$, $\mathcal{M} \in \mathbb{R}^{2n \times 2n}$, $\mathcal{Q}_1 \in \mathbb{R}^{2n \times 2n}$, $\mathcal{Q}_2 \in \mathbb{R}^{2n \times 2n}$, $Q_3 \in \mathbb{R}^{n \times n}$, $Q_4 \in \mathbb{R}^{n \times n}$, and any matrix $\mathcal{S} \in \mathbb{R}^{2n \times 2n}$ satisfying the following LMIs:*

$$(\Upsilon^{\perp})^T (\Phi + \Psi)(\Upsilon^{\perp}) < 0_{9n}, \tag{11}$$

$$\begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix} \ge 0_{4n}, \tag{12}$$

*where $\Phi$, $\Upsilon$ are defined in Eq. (10), and $\Upsilon^{\perp}$ is the right orthogonal complement of $\Upsilon$.*

**Proof.** Define the forward difference of $V(k)$ as

$$\Delta V(k) = V(k+1) - V(k). \tag{13}$$

Let us consider the following Lyapunov–Krasovskii functional candidate as

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{14}$$

where

$$V_1(k) = \alpha^T(k) \mathcal{R} \alpha(k),$$

$$V_2(k)=\sum_{s=k-h_m}^{k-1}\beta^T(s)\mathcal{N}\beta(s)+\sum_{s=k-h_M}^{k-h_m-1}\beta^T(s)\mathcal{M}\beta(s),$$

$$V_3(k)=h_m\sum_{s=-h_m}^{-1}\sum_{u=k+s}^{k-1}\beta^T(u)\mathcal{Q}_1\beta(u)+(h_M-h_m)\sum_{s=-h_M}^{-h_m-1}\sum_{u=k+s}^{k-h_m-1}\beta^T(u)\mathcal{Q}_2\beta(u),$$

$$\begin{aligned}V_4(k)=&\left(\frac{h_m(h_m+1)}{2}\right)\sum_{s=-h_m}^{-1}\sum_{u=s}^{-1}\sum_{v=k+u}^{k-1}\Delta x^T(v)Q_3\Delta x(v)\\&+\left(\left(\frac{h_M-h_m}{2}\right)(h_M-h_m+1)\right)\sum_{s=-h_M}^{-h_m-1}\sum_{u=s}^{-1}\sum_{v=k+u}^{k-h_m-1}\Delta x^T(v)Q_4\Delta x(v).\end{aligned} \tag{15}$$

The forward difference of $V_1(k)$ is calculated as

$$\Delta V_1(k)=\alpha^T(k+1)\mathcal{R}\alpha(k+1)-\alpha^T(k)\mathcal{R}\alpha(k). \tag{16}$$

Here, it can be confirmed the following equations:

$$\alpha(k+1)=\begin{bmatrix}x(k+1)\\x(k+1-h_m)\\x(k+1-h_M)\\\sum_{s=k+1-h_m}^{k}x(s)\\\sum_{s=k+1-h_M}^{k-h_m}x(s)\end{bmatrix}=\begin{bmatrix}x(k)+\Delta x(k)\\x(k-h_m)+\Delta x(k-h_m)\\x(k-h_M)+\Delta x(k-h_M)\\\begin{pmatrix}x(k)-x(k-h_m)\\+\sum_{s=k-h_m}^{k-1}x(s)\end{pmatrix}\\\begin{pmatrix}x(k-h_m)-x(k-h_M)\\+\sum_{s=k-h(k)}^{k-h_m-1}x(s)\\+\sum_{s=k-h_M}^{k-h(k)-1}x(s)\end{pmatrix}\end{bmatrix}=\Pi_1^T\zeta(k), \tag{17}$$

$$\alpha(k)=\begin{bmatrix}x(k)\\x(k-h_m)\\x(k-h_M)\\\sum_{s=k-h_m}^{k}x(s)\\\begin{pmatrix}\sum_{s=k-h(k)}^{k-h_m-1}x(s)\\+\sum_{s=k-h_M}^{k-h(k)-1}x(s)\end{pmatrix}\end{bmatrix}=\Pi_2^T\zeta(k). \tag{18}$$

Thus, $\Delta V_1(k)$ can be represented as

$$\Delta V_1(k)=\zeta^T(k)[\Pi_1\mathcal{R}\Pi_1^T-\Pi_2\mathcal{R}\Pi_2^T]\zeta(k)=\zeta^T(k)\Xi_1\zeta(k). \tag{19}$$

The forward difference of $V_2(k)$ is obtained as

$$\begin{aligned}\Delta V_2(k)&=\beta^T(k)\mathcal{N}\beta(k)-\beta^T(k-h_m)\mathcal{N}\beta(k-h_m)\\&\quad+\beta^T(k-h_m)\mathcal{M}\beta(k-h_m)-\beta^T(k-h_M)\mathcal{M}\beta(k-h_M)\\&=\zeta^T(k)\left(\Pi_3\mathcal{N}\Pi_3^T+\Pi_4(-\mathcal{N}+\mathcal{M})\Pi_4^T-\Pi_5 M\Pi_5^T\right)\zeta(k)\\&=\zeta^T(k)\Xi_2\zeta(k).\end{aligned} \tag{20}$$

Calculating $\Delta V_3(k)$ gives

$$\Delta V_3(k) = (h_m)^2 \beta^T(k)\mathcal{Q}_1\beta(k) - h_m \sum_{s=k-h_m}^{k-1} \beta^T(s)\mathcal{Q}_1\beta(s) + (h_M-h_m)^2\beta^T(k-h_m)\mathcal{Q}_2\beta(k-h_m) - (h_M-h_m)\sum_{s=k-h_M}^{k-h_m-1}\beta^T(s)\mathcal{Q}_2\beta(s). \tag{21}$$

Here, by utilizing Eq. (2) of Lemma 1, it can be obtained that

$$\begin{aligned} -h_m \sum_{s=k-h_m}^{k-1} \beta^T(s)\mathcal{Q}_1\beta(s) &= -h_m \sum_{s=k-h_m}^{k-1} \begin{bmatrix} x(s) \\ \Delta x(s) \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} x(s) \\ \Delta x(s) \end{bmatrix} \\ &\le -\left( \sum_{s=k-h_m}^{k-1} \begin{bmatrix} x(s) \\ \Delta x(s) \end{bmatrix} \right)^T \mathcal{Q}_1 \left( \sum_{s=k-h_m}^{k-1} \begin{bmatrix} x(s) \\ \Delta x(s) \end{bmatrix} \right) \\ &= -\begin{bmatrix} \sum_{s=k-h_m}^{k-1} x(s) \\ x(k)-x(k-h_m) \end{bmatrix}^T \mathcal{Q}_1 \begin{bmatrix} \sum_{s=k-h_m}^{k-1} x(s) \\ x(k)-x(k-h_m) \end{bmatrix}. \end{aligned} \tag{22}$$

When $h_m < h(k) < h_M$, we have

$$\begin{aligned} &-(h_M-h_m)\sum_{s=k-h_M}^{k-h_m-1} \beta^T(s)\mathcal{Q}_2\beta(s) \\ &\quad = -(h_M-h_m)\sum_{s=k-h(k)}^{k-h_m-1} \begin{bmatrix} x(s) \\ \Delta x(s) \end{bmatrix}^T \mathcal{Q}_2 \begin{bmatrix} x(s) \\ \Delta x(s) \end{bmatrix} \\ &\qquad -(h_M-h_m)\sum_{s=k-h_M}^{k-h(k)-1} \begin{bmatrix} x(s) \\ \Delta x(s) \end{bmatrix}^T \mathcal{Q}_2 \begin{bmatrix} x(s) \\ \Delta x(s) \end{bmatrix} \\ &\quad \le -\begin{bmatrix} \sum_{s=k-h(k)}^{k-1} x(s) \\ x(k-h_m)-x(k-h(k)) \\ \sum_{s=k-h_M}^{k-h(k)-1} x(s) \\ x(k-h(k))-x(k-h_M) \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}_2/(1-\alpha_k) & 0_{2n} \\ \star & \mathcal{Q}_2/\alpha_k \end{bmatrix} \begin{bmatrix} \sum_{s=k-h(k)}^{k-1} x(s) \\ x(k-h_m)-x(k-h(k)) \\ \sum_{s=k-h_M}^{k-h(k)-1} x(s) \\ x(k-h(k))-x(k-h_M) \end{bmatrix}, \end{aligned} \tag{23}$$

where $\alpha_k = (h_M-h(k))(h_M-h_m)^{-1}$ which satisfies $0 < \alpha_k < 1$.

By reciprocally convex approach [21], if the inequality (12) holds, then the following inequality holds

$$\begin{bmatrix} \sqrt{\frac{\alpha_k}{1-\alpha_k}} I_{2n} & 0_{2n} \\ \star & -\sqrt{\frac{1-\alpha_k}{\alpha_k}} I_{2n} \end{bmatrix} \begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\alpha_k}{1-\alpha_k}} I_{2n} & 0_{2n} \\ \star & -\sqrt{\frac{1-\alpha_k}{\alpha_k}} I_{2n} \end{bmatrix} \ge 0_{4n}, \tag{24}$$

which implies

$$\begin{bmatrix} \mathcal{Q}_2/(1-\alpha_k) & 0_{2n} \\ \star & \mathcal{Q}_2/\alpha_k \end{bmatrix} \geq \begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix}. \tag{25}$$

Thus, from Eqs. (23) and (25), we have

$$\begin{aligned} &-(h_M-h_m)\sum\nolimits_{s=k-h_M}^{k-h_m-1} \beta^T(s)\mathcal{Q}_2\beta(s) \\ &\leq -\begin{bmatrix} \sum\nolimits_{s=k-h(k)}^{k-h_m-1} x(s) \\ x(k-h_m)-x(k-h(k)) \\ \sum\nolimits_{s=k-h_M}^{k-h(k)-1} x(s) \\ x(k-h(k))-x(k-h_M) \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix} \begin{bmatrix} \sum\nolimits_{s=k-h(k)}^{k-h_m-1} x(s) \\ x(k-h_m)-x(k-h(k)) \\ \sum\nolimits_{s=k-h_M}^{k-h(k)-1} x(s) \\ x(k-h(k))-x(k-h_M) \end{bmatrix}. \end{aligned} \tag{26}$$

It should be noted that when $h(k)=h_m$ or $h(k)=h_M$, we have $x(k-h_m)-x(k-h(k))=0$ or $x(k-h(k))-x(k-h_M)=0$, respectively. Thus, Eq. (26) still holds. Therefore, an upper bound of $\Delta V_3(k)$ can be represented as

$$\begin{aligned} \Delta V_3(k) &\leq \zeta^T(k)\left[(h_m^2)\Pi_3\mathcal{Q}_1\Pi_3^T + (h_M-h_m)^2\Pi_4\mathcal{Q}_2\Pi_4^T - \Pi_6\mathcal{Q}_2\Pi_6^T - \Pi_7\begin{bmatrix} \mathcal{Q}_2 & \mathcal{S} \\ \star & \mathcal{Q}_2 \end{bmatrix}\Pi_7^T\right]\zeta(k) \\ &= \zeta^T(k)(\Xi_3+\Psi)\zeta(k). \end{aligned} \tag{27}$$

The calculation of $\Delta V_4(k)$ leads to

$$\begin{aligned} \Delta V_4(k) &= \left(\frac{h_m(h_m+1)}{2}\right)\sum_{s=-h_m}^{-1}\sum_{u=s}^{-1}\left(\sum_{v=k+u+1}^{k}\Delta x^T(v)Q_3\Delta x(v) - \sum_{v=k+u}^{k-1}\Delta x^T(v)Q_3\Delta x(v)\right) \\ &\quad + \left(\left(\frac{h_M-h_m}{2}\right)(h_M-h_m+1)\right) \\ &\quad \times \sum_{s=-h_M}^{-h_m-1}\sum_{u=s}^{-h_m-1}\left(\sum_{v=k+u+1}^{k-h_m}\Delta x^T(v)Q_4\Delta x(v) - \sum_{v=k+u}^{k-h_m-1}\Delta x^T(v)Q_4\Delta x(v)\right) \\ &= \left(\frac{h_m(h_m+1)}{2}\right)\sum_{s=-h_m}^{-1}\sum_{u=s}^{-1}\left(\Delta x^T(k)Q_3\Delta x(k) - \Delta x^T(k+u)Q_3\Delta x(k+u)\right) \\ &\quad + \left(\left(\frac{h_M-h_m}{2}\right)(h_M-h_m+1)\right) \\ &\quad \times \sum_{s=-h_M}^{-h_m-1}\sum_{u=s}^{-h_m-1}\left(\Delta x^T(k-h_m)Q_4\Delta x(k-h_m) - \Delta x^T(k+u)Q_4\Delta x(k+u)\right) \\ &= \left(\frac{h_m(h_m+1)}{2}\right)\sum_{s=-h_m}^{-1}\left((-s)\Delta x^T(k)Q_3\Delta x(k) - \sum_{u=s}^{-1}\Delta x^T(k+u)Q_3\Delta x(k+u)\right) \\ &\quad + \left(\left(\frac{h_M-h_m}{2}\right)(h_M-h_m+1)\right) \\ &\quad \times \sum_{s=-h_M}^{-h_m-1}\left((-s-h_m)\Delta x^T(k-h_m)Q_4\Delta x(k-h_m) - \sum_{u=s}^{-h_m-1}\Delta x^T(k+u)Q_4\Delta x(k+u)\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{h_m(h_m+1)}{2}\right)^2 \Delta x^T(k) Q_3 \Delta x(k) \\
&\quad -\left(\frac{h_m(h_m+1)}{2}\right) \sum_{s=-h_m}^{-1} \sum_{u=s}^{-1} \Delta x(k+u) Q_3 \Delta x(k+u) \\
&\quad +\left(\left(\frac{h_M-h_m}{2}\right)(h_M-h_m+1)\right)^2 \Delta x^T(k-h_m) Q_4 \Delta x(k-h_m) \\
&\quad -\left(\left(\frac{h_M-h_m}{2}\right)(h_M-h_m+1)\right) \sum_{s=-h_M}^{-h_m-1} \sum_{u=s}^{-h_m-1} \Delta x^T(k+u) Q_4 \Delta x(k+u).
\end{aligned} \tag{28}
$$

By utilizing Eq. (2) of Lemma 1, it should be noted that

$$
\begin{aligned}
&-\left(\frac{h_m(h_m+1)}{2}\right) \sum_{s=-h_m}^{-1} \sum_{u=s}^{-1} \Delta x(k+u) Q_3 \Delta x(k+u) \\
&\quad \le -\left(\sum_{s=-h_m}^{-1} \Delta x(k+u)\right)^T Q_3 \left(\sum_{s=-h_m}^{-1} \Delta x(k+u)\right) \\
&\quad = -\left(\sum_{s=-h_m}^{-1} (x(k)-x(k+s))\right)^T Q_3 \left(\sum_{s=-h_m}^{-1} (x(k)-x(k+s))\right) \\
&\quad = -\left(h_m x(k) - \sum_{s=k-h_m}^{k-1} x(s)\right)^T Q_3 \left(h_m x(k) - \sum_{s=k-h_m}^{k-1} x(s)\right) \\
&\quad = -\zeta^T(k)\left[(h_m e_1 - e_8) Q_3 (h_m e_1 - e_8)^T\right]\zeta(k),
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
&-\left(\left(\frac{h_M-h_m}{2}\right)(h_M-h_m+1)\right) \sum_{s=-h_M}^{-h_m-1} \sum_{u=s}^{-h_m-1} \Delta x^T(k+u) Q_4 \Delta x(k+u) \\
&\quad \le -\left(\sum_{s=-h_M}^{-h_m-1} \sum_{u=k+s}^{k-h_m-1} \Delta x(u)\right)^T Q_4 \left(\sum_{s=-h_M}^{-h_m-1} \sum_{u=k+s}^{k-h_m-1} \Delta x(u)\right) \\
&\quad = -\left(\sum_{s=-h_M}^{-h_m-1} (x(k-h_m)-x(k+s))\right)^T Q_4 \left(\sum_{s=-h_M}^{-h_m-1} (x(k-h_m)-x(k+s))\right) \\
&\quad = -\left((h_M-h_m)x(k-h_m) - \sum_{s=k-h_M}^{k-h_m-1} x(s)\right)^T Q_4 \left((h_M-h_m)x(k-h_m) - \sum_{s=k-h_M}^{k-h_m-1} x(s)\right) \\
&\quad = -\left((h_M-h_m)x(k-h_m) - \sum_{s=k-h(k)}^{k-h_m-1} x(s) - \sum_{s=k-h_M}^{k-h(k)-1} x(s)\right)^T Q_4 \\
&\qquad \times \left((h_M-h_m)x(k-h_m) - \sum_{s=k-h(k)}^{k-h_m-1} x(s) - \sum_{s=k-h_M}^{k-h(k)-1} x(s)\right) \\
&\quad = -\zeta^T(k)\left[((h_M-h_m)e_2 - e_9 - e_{10}) Q_4 ((h_M-h_m)e_2 - e_9 - e_{10})^T\right]\zeta(k).
\end{aligned} \tag{30}
$$

Therefore, from Eqs. (29) and (30), an upper bound of $\Delta V_4(k)$ can be

$$\Delta V_4(k) \leq \zeta^T(k)\Xi_4\zeta(k). \tag{31}$$

From Eqs. (13) to (31), $\Delta V(k)$ has a new upper bound as

$$\Delta V(k) \leq \zeta^T(k)\left(\sum_{i=1}^{4}\Xi_i + \Psi\right)\zeta(k) \equiv \zeta^T(k)(\Phi + \Psi)\zeta(k). \tag{32}$$

Also, the system (1) with the augmented matrix $\zeta(k)$ can be rewritten as $\Upsilon\zeta(k) = 0_{n\times 1}$. Then, a delay-dependent stability condition for the system (1) is

$$\zeta^T(k)(\Phi + \Psi)\zeta(k) < 0 \quad \text{subject to} \quad \Upsilon\zeta(k) = 0_{n\times 1}. \tag{33}$$

By Lemma 2, the condition (33) is equivalent to the inequality (11). Therefore, if the inequality (11) and (12) hold, then the system (1) is asymptotically stable by Lyapunov stability. This completes our proof. □

**Remark 1.** In the field of delay-dependent stability analysis, one of the major concerns is to get maximum delay bounds with fewer decision variables [7]. From Lemma 2, one can check that ${\Upsilon^{\perp}}^T\Phi\Upsilon^{\perp} < 0$ is equivalent to the existence of $\mathcal{X}$ such that $\Phi + \mathcal{X}\Upsilon + \Upsilon^T\mathcal{X}^T < 0$ holds. Any matrix $\mathcal{X}$ of the condition (iii) of Lemma 2 plays a role as a free-weighting matrices $\mathcal{X} = [X_1^T, X_2^T, \ldots, X_{10}^T]^T$ of zero equality $2\zeta^T(k)\mathcal{X}\Upsilon\zeta(k) = 0$. However, the condition (iii) of Lemma 2 has and disadvantage of more decision variables than the condition (ii) of Lemma 2. Therefore, our proposed stability criteria are derived in the form of (ii) in Lemma 2.

**Remark 2.** In [9–14], the following equation in the form of double summation

$$\sum_{s=-h_M}^{-h_m-1}\sum_{u=k+s}^{k-1}\Delta x^T(u)\mathcal{R}\Delta x(u), \tag{34}$$

were utilized in the Lyapunov–Krasovskii functional. In Eq. (34), we set the interval of double summation $-h_M \leq s \leq -h_m - 1$ and $k + s \leq u \leq k - 1$. Since the parameter $s$ has the interval from $-h_M$ to $-h_m - 1$, it may be effective to reduce the conservatism of stability criteria if the maximum value of $u$ is changed as $k - h_m - 1$ instead of $k - 1$. In the continuous stability criteria for systems with interval time-varying delays, one can confirm this fact in [26] and [27]. With this regard, in Theorem 1, the terms

$$(h_M - h_m)\sum_{s=-h_M}^{-h_m-1}\sum_{u=k+s}^{k-h_m-1}\beta^T(u)Q_2\beta(u), \tag{35}$$

and

$$\left(\left(\frac{h_M - h_m}{2}\right)(h_M - h_m + 1)\right)\sum_{s=-h_M}^{-h_m-1}\sum_{u=s}^{-1}\sum_{v=k+u}^{k-h_m-1}\Delta x^T(v)Q_4\Delta x(v), \tag{36}$$

were proposed for the first time at each $V_3$ and $V_4$, respectively.

**Remark 3.** In the proposed Laypunov–Krasovsikii functional (14), the utilized augmented vector includes the terms such as $\sum_{s=k-h_m}^{k-1}x(s)$, $\sum_{s=k-h(k)}^{k-h_m-1}x(s)$, and $\sum_{s=k-h_M}^{k-h(k)-1}x(s)$. By these terms, more past history of $x(k)$ can be utilized, which may lead less conservative results. Thus, $V_1(k)$ and $V_3(k)$ in which the summation terms can obtained by calculating $\Delta V_1(k)$ and $\Delta V_3(k)$ are proposed in this paper.

Based on Theorem 1, a further improved delay-dependent stability criterion of the system (1) is given by the following theorem.

**Theorem 2.** *For given integers $h_m$, $h_M$ satisfying $0<h_m<h_M$ and a controller gain K, the system (1) is asymptotically stable for $h_m \le h(k) \le h_M$, if there exist positive definite matrices $\mathcal{R} \in \mathbb{R}^{5n\times 5n}$, $\mathcal{N} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{M} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_1 \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Q}_2 \in \mathbb{R}^{2n\times 2n}$, $Q_3 \in \mathbb{R}^{n\times n}$, $Q_4 \in \mathbb{R}^{n\times n}$, any symmetric matrices $P_i \in \mathbb{R}^{n\times n} (i=1,2,3)$, and any matrix $\mathcal{S} \in \mathbb{R}^{2n\times 2n}$ satisfying the following LMIs:*

$$(\Upsilon^{\perp})^T(\Phi + \tilde{\Psi} + \Omega)(\Upsilon^{\perp}) < 0_{9n}, \tag{37}$$

$$\mathcal{Q}_1 + \begin{bmatrix} 0_n & P_1 \\ \star & P_1 \end{bmatrix} > 0_{2n}, \tag{38}$$

$$\left[\begin{array}{c|c} \mathcal{Q}_2 + \begin{bmatrix} 0_n & P_2 \\ \star & P_2 \end{bmatrix} & \mathcal{S} \\ \hline \star & \mathcal{Q}_2 + \begin{bmatrix} 0_n & P_3 \\ \star & P_3 \end{bmatrix} \end{array}\right] \ge 0_{4n}, \tag{39}$$

*where $\Phi$, $\Upsilon$ are defined in Eq. (10), $\Upsilon^{\perp}$ is the right orthogonal complement of $\Upsilon$,*

$$\tilde{\Psi} = -\Pi_6\left(\mathcal{Q}_1 + \begin{bmatrix} 0_n & P_1 \\ \star & P_1 \end{bmatrix}\right)\Pi_6^T - \Pi_7 \left[\begin{array}{c|c} \mathcal{Q}_2 + \begin{bmatrix} 0_n & P_2 \\ \star & P_2 \end{bmatrix} & \mathcal{S} \\ \hline \star & \mathcal{Q}_2 + \begin{bmatrix} 0_n & P_3 \\ \star & P_3 \end{bmatrix} \end{array}\right] \Pi_7^T, \tag{40}$$

*and*

$$\Omega = h_m e_1 P_1 e_1^T - h_m e_2 P_1 e_2^T + (h_M - h_m)(e_2 P_2 e_2 + e_3(-P_2 + P_3)e_3^T - e_4 P_3 e_4^T), \tag{41}$$

**Proof.** Let us consider the same Lyapunov–Krasovskii functional introduced in Eq. (14). For any matrix $P$, integers $l_1$ and $l_2$ satisfying $l_1 < l_2$, and vector function $x(s) : [k-l_2, k-l_1-1] \to \mathbb{R}^n$, the following equality holds:

$$x^T(k-l_1)Px(k-l_1) - x^T(k-l_2)Px(k-l_2) = \sum_{s=k-l_2}^{k-l_1-1} (x^T(s+1)Px(s+1) - x^T(s)Px(s)). \tag{42}$$

It should be noted that

$$\begin{aligned} &x^T(s+1)Px(s+1) - x^T(s)Px(s) \\ &\quad = (\Delta x(s) + x(s))^T P(\Delta x(s) + x(s)) - x^T(s)Px(s) \\ &\quad = \Delta x^T(s)P\Delta x(s) + 2x^T(s)P\Delta x(s). \end{aligned} \tag{43}$$

From the equalities (42) and (43), by choosing $(l_1, l_2)$ as $(0, h_m)$, $(h_m, h(k))$, $(h(k), h_M)$, respectively, the following three zero equalities with any symmetric matrices $P_i (i=1,2,3)$ hold:

$$0 = x^T(k)(h_m P_1)x(k) - x^T(k-h_m)(h_m P_1)x(k-h_m)$$

$$-h_m \sum_{s=k-h_m}^{k-1} (\Delta x^T(s)P_1\Delta x(s) + 2x^T(s)P_1\Delta x(s)), \tag{44}$$

$$\begin{aligned} 0 = {} & x^T(k-h_m)((h_M-h_m)P_2)x(k-h_m) - x^T(k-h(k))((h_M-h_m)P_2)x(k-h(k)) \\ & -(h_M-h_m) \sum_{s=k-h(k)}^{k-h_m-1} (\Delta x^T(s)P_2\Delta x(s) + 2x^T(s)P_2\Delta x(s)), \end{aligned} \tag{45}$$

$$\begin{aligned} 0 = {} & x^T(k-h(k))((h_M-h_m)P_3)x(k-h(k)) - x^T(k-h_M)((h_M-h_m)P_3)x(k-h_M) \\ & -(h_M-h_m) \sum_{s=k-h_M}^{k-h(k)-1} \left(\Delta x^T(s)P_3\Delta x(s) + 2x^T(s)P_3\Delta x(s)\right). \end{aligned} \tag{46}$$

By adding the above three zero equalities to Eq. (21) and using the similar method shown in Eq. (22) and (23), if (38) and (39) hold, then a new upper bound of $\Delta V_3(k)$ can be obtained as

$$\begin{aligned} \Delta V_3(k) = {} & x^T(k)(h_mP_1)x(k) - x^T(k-h_m)(h_mP_1)x(k-h_m) \\ & + x^T(k-h_m)((h_M-h_m)P_2)x(k-h_m) - x^T(k-h(k))((h_M-h_m)P_2)x(k-h(k)) \\ & + x^T(k-h(k))((h_M-h_m)P_3)x(k-h(k)) - x^T(k-h_M)((h_M-h_m)P_3)x(k-h_M) \\ & + (h_m)^2\beta^T(k)\mathcal{Q}_1\beta(k) + (h_M-h_m)^2\beta^T(k-h_m)\mathcal{Q}_2\beta(k) \\ & - h_m \sum_{s=k-h_m}^{k-1} \beta^T(s)\left(\mathcal{Q}_1 + \begin{bmatrix} 0_n & P_1 \\ \star & P_1 \end{bmatrix}\right)\beta(s) \\ & -(h_M-h_m) \sum_{s=k-h(k)}^{k-h_m-1} \beta^T(s)\left(\mathcal{Q}_2 + \begin{bmatrix} 0_n & P_2 \\ \star & P_2 \end{bmatrix}\right)\beta(s) \\ & -(h_M-h_m) \sum_{s=k-h_M}^{k-(k)-1} \beta^T(s)\left(\mathcal{Q}_2 + \begin{bmatrix} 0_n & P_3 \\ \star & P_3 \end{bmatrix}\right)\beta(s) \\ \le {} & \zeta^T(k)(\Xi_3 + \tilde{\Psi} + \Omega)\zeta(k). \end{aligned} \tag{47}$$

The other procedure is straightforward from the proof of Theorem 1, so it is omitted. □

**Remark 4.** In Theorem 2, motivated by the work [18], three zero equalities (44)–(46) were proposed and utilized to reduce the conservatism of the stability condition presented in Theorem 1. As presented in Eqs. (44)–(46), the terms $x^T(k)(h_mP_1)x(k) - x^T(k-h_m)(h_mP_1)x(k-h_m)$, $x^T(k-h_m)((h_M-h_m)P_2)x(k-h_m) - x^T(k-h(k))((h_M-h_m)P_2)x(k-h(k))$, and $x^T(k-h(k))((h_M-h_m)P_3)x(k-h(k)) - x^T(k-h_M)((h_M-h_m)P_3)x(k-h_M)$ provide the enhanced feasible region of the stability. Furthermore, as shown in Eq. (47), the three summation terms in Eqs. (44)–(46) are merged into $-h_m\sum_{s=k-h_m}^{k-1}\beta^T(s)\mathcal{Q}_1\beta(s)$, $-(h_M-h_m)\sum_{s=k-h(k)}^{k-h_m-1}\beta^T(s)\mathcal{Q}_2\beta(s)$, and $-(h_M-h_m)\sum_{s=k-h_M}^{k-(k)-1}\beta^T(s)\mathcal{Q}_2\beta(s)$, which cause the conservatism in estimating $\Delta V_3(k)$. Therefore, comparing with the Eqs. (22) and (23), the obtained estimation of $\Delta V_3(k)$ in Eq. (47) has more flexibility due to the existence of $P_i$. Without loss of generality, if $P_i (i=1,2,3)$ in Theorem 2 are zero matrices, then the corresponding stability condition is equal to the stability criterion presented in Theorem 1 since the same Lyapunov–Krasovskii functional and augmented vector $\zeta(k)$ of Theorem 2 are used in Theorem 1. Through one numerical example, the enhancement of feasible region of Theorem 2 comparing with those of Theorem 1 will be shown by investigating maximum delay bounds.

Finally, based on the result of Theorem 2, a designing method of time-delayed controller gain $K$ will be proposed in Theorem 3. For simplicity, the notations which will be used in Theorem 3 are defined as

$$\hat{\Xi}_1 = \Pi_1 \hat{\mathcal{R}} \Pi_1^T - \Pi_2 \hat{\mathcal{R}} \Pi_2^T, \hat{\Xi}_2 = \Pi_3 \hat{\mathcal{N}} \Pi_3^T + \Pi_4(-\hat{\mathcal{N}} + \hat{\mathcal{M}})\Pi_4^T - \Pi_5 \hat{\mathcal{M}} \Pi_5^T,$$
$$\hat{\Xi}_3 = (h_m^2)\Pi_3 \hat{\mathcal{Q}}_1 \Pi_3^T + (h_M - h_m)^2 \Pi_4 \hat{\mathcal{Q}}_2 \Pi_4^T,$$
$$\hat{\Xi}_4 = \left(\frac{h_m(h_m+1)}{2}\right)^2 e_5 \hat{Q}_3 e_5^T + \left(\left(\frac{h_M - h_m}{2}\right)(h_M - h_m + 1)\right)^2 e_6 \hat{Q}_3 e_6^T,$$
$$-(h_m e_1 - e_8)\hat{Q}_3 (h_m e_1 - e_8)^T - ((h_M - h_m)e_2 - e_9 - e_{10})\hat{Q}_4((h_M - h_m)e_2 - e_9 - e_{10})^T,$$
$$\hat{\Phi} = \sum_{i=1}^{4} \hat{\Xi}_i. \tag{48}$$

Now, we have the following theorem.

**Theorem 3.** *For given integers $0 < h_m < h_M$ and a scalar value $\delta$, the system (1) under the controller $u(k) = YX^{-1}x(k - h(k))$ is asymptotically stabilized for $h_m \le h(k) \le h_M$, if there exist positive definite matrices $\hat{\mathcal{R}} \in \mathbb{R}^{5n \times 5n}$, $\hat{\mathcal{N}} \in \mathbb{R}^{2n \times 2n}$, $\hat{\mathcal{M}} \in \mathbb{R}^{2n \times 2n}$, $\hat{\mathcal{Q}}_1 \in \mathbb{R}^{2n \times 2n}$, $\hat{\mathcal{Q}}_2 \in \mathbb{R}^{2n \times 2n}$, $\hat{Q}_3 \in \mathbb{R}^{n \times n}$, $\hat{Q}_4 \in \mathbb{R}^{n \times n}$, any symmetric matrices $\hat{P}_i \in \mathbb{R}^{n \times n} (i = 1,2,3)$, any matrix $\hat{\mathcal{S}} \in \mathbb{R}^{2n \times 2n}$, $X \in \mathbb{R}^{n \times n}$, and $Y \in \mathbb{R}^{m \times n}$ satisfying the following LMIs:*

$$\hat{\Phi} + \tilde{\Psi} + \hat{\Omega} + \hat{\Gamma}\hat{\Lambda} + \hat{\Lambda}^T \hat{\Gamma}^T < 0, \tag{49}$$

$$\hat{\mathcal{Q}}_1 + \begin{bmatrix} 0_n & \hat{P}_1 \\ \star & \hat{P}_1 \end{bmatrix} > 0, \tag{50}$$

$$\left[\begin{array}{c|c} \hat{\mathcal{Q}}_2 + \begin{bmatrix} 0_n & \hat{P}_2 \\ \star & \hat{P}_2 \end{bmatrix} & \hat{\mathcal{S}} \\ \hline \star & \hat{\mathcal{Q}}_2 + \begin{bmatrix} 0_n & \hat{P}_3 \\ \star & \hat{P}_3 \end{bmatrix} \end{array}\right] \ge 0, \tag{51}$$

*where $\hat{\Phi}$ is defined in (48),*

$$\tilde{\Psi} = -\Pi_6\left(\hat{\mathcal{Q}}_1 + \begin{bmatrix} 0_n & \hat{P}_1 \\ \star & \hat{P}_1 \end{bmatrix}\right)\Pi_6^T - \Pi_7 \left[\begin{array}{c|c} \hat{\mathcal{Q}}_2 + \begin{bmatrix} 0_n & \hat{P}_2 \\ \star & \hat{P}_2 \end{bmatrix} & \hat{\mathcal{S}} \\ \hline \star & \hat{\mathcal{Q}}_2 + \begin{bmatrix} 0_n & \hat{P}_3 \\ \star & \hat{P}_3 \end{bmatrix} \end{array}\right] \Pi_7^T,$$
$$\hat{\Omega} = h_m e_1 \hat{P}_1 e_1^T - h_m e_2 \hat{P}_1 e_2^T + (h_M - h_m)(e_2 \hat{P}_2 e_2 + e_3(-\hat{P}_2 + \hat{P}_3)e_3^T - e_4 \hat{P}_3 e_4^T),$$
$$\hat{\Gamma} = [e_1 X + e_5(\delta X)], \tag{52}$$

*and*

$$\hat{\Lambda} = [(A - I_n)X, 0_n, BY, 0_n, -X, 0_n, 0_n, 0_n, 0_n, 0_n]. \tag{53}$$

**Proof.** Let us define

$$\Gamma = [e_1 F + e_5(\delta F)], \tag{54}$$

and

$$\Lambda = [(A - I_n), 0_n, BK, 0_n, -I_n, 0_n, 0_n, 0_n, 0_n, 0_n]. \tag{55}$$

Then, the following zero equation holds for any matrix $F$:

$$\begin{aligned} 0 &= 2[x^T(k)F + \Delta x^T(k)(\delta F)][(A - I_n)x(k) + BKx(k - h(k)) - \Delta x(k)] \\ &= \zeta^T(k)[\Gamma\Lambda + \Lambda^T\Gamma^T]\zeta(k). \end{aligned} \tag{56}$$

With the same Lyapunov–Krasovskii functional candidate in Eq. (14) and considering Eq. (56), by using the similar method presented in the proofs of Theorems 1 and 2, a sufficient condition guaranteeing the asymptotic stability for the system (1) can be

$$\Phi + \tilde{\Psi} + \Omega + \Gamma\Lambda + \Lambda^T\Gamma^T < 0, \tag{57}$$

where $\Phi$, $\tilde{\Psi}$, and $\Omega$ are the same notations used in Theorem 2. Let us define

$$\begin{aligned} &X = F^{-1}, \\ &\hat{\mathcal{R}} = diag\{X,X,X,X,X\}^T \mathcal{R}\, diag\{X,X,X,X,X\}, \\ &\hat{\mathcal{N}} = diag\{X,X\}^T \mathcal{N}\, diag\{X,X\}, \\ &\hat{\mathcal{M}} = diag\{X,X\}^T \mathcal{M}\, diag\{X,X\}, \\ &\hat{\mathcal{Q}}_1 = diag\{X,X\}^T \mathcal{Q}_1\, diag\{X,X\}, \\ &\hat{\mathcal{Q}}_2 = diag\{X,X\}^T \mathcal{Q}_2\, diag\{X,X\}, \\ &\hat{Q}_3 = X^T Q_3 X, \hat{Q}_4 = X^T Q_4 X, \\ &\hat{\mathcal{S}} = diag\{X,X\}^T \mathcal{S}\, diag\{X,X\}, \\ &\hat{\mathcal{P}}_1 = X^T \mathcal{P}_1 X, \\ &\hat{\mathcal{P}}_2 = X^T \mathcal{P}_2 X, \\ &\hat{\mathcal{P}}_3 = X^T \mathcal{P}_3 X, \end{aligned} \tag{58}$$

and $Y = KX$. Then, pre- and post-multiplying inequality (57) by matrices $diag\,\{X,X,X,X,X,X,X,X,X,X\}^T$ and $diag\{X,X,X,X,X,X,X,X,X,X\}$ leads to LMIs (49). Also, the conditions (38) and (39) in Theorem 2 are changed into the two inequalities (50) and (51). This completes our proof. □

## 4. Numerical examples

In this section, we provide three numerical examples to illustrate the effectiveness of the stability and stabilization criteria developed by this paper.

**Example 1.** Consider the following system:

$$x(k+1) = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} x(k - h(k)), \tag{59}$$

which was used to check the feasible region of stability criteria in [8–11], [13,14]. For various low bound of $h(k)$, the comparison of the obtained results by applying Theorems 1 and 2 to the above system is conducted as listed in Table 1. From Table 1, it can be confirmed that Theorem 1 provides larger feasible region than those of [8–11] and [13,14] except for the case of $h_m = 0$. However, it can be seen that the results obtained by Theorem 2 clearly reduce the conservatism of Theorem 1 by utilizing three zero equalities proposed in Eqs. (44)–(46).

**Remark 5.** Recently, the delay-partitioning approaches to enhance the feasible region of discrete-time system with interval time-varying delays were proposed in [12] for the first time. The advantage of the delay-partitioning approaches can obtain more tighter upper bound of summation terms obtained by calculating $\Delta V(k)$ by dividing delay intervals into some subintervals. However, when delay-partitioning number increases, the decision number also increases. Furthermore, matrix formulation becomes more complex and the computational burden and time-consuming grow bigger. In Table 2, maximum delay bounds and number of decision variables are compared with the results of [9] and [12]. Table 2 shows that the obtained results of Theorem 2 provides larger delay bounds than all the results of [9] and [12], although the decision variables of Theorem 1 and 2 are smaller than those of [9] and [12].

**Example 2.** Consider the following model of inverted pendulum described in [9] and shown in Fig. 1

$$\dot{x}(t)=\begin{bmatrix} 0 & 1 \\ \dfrac{3(M+m)g}{l(4M+m)} & 0 \end{bmatrix} x(t)+\begin{bmatrix} 0 \\ -\dfrac{3}{l(4M+m)} \end{bmatrix} u(t). \tag{60}$$

By choosing $M=8$ kg, $m=2.0$ kg, $l=0.5$ m, $g=9.8\ \mathrm{m/s^2}$ and sampling time $T_s=30$ ms, the continuous-time system (60) can be transformed as the following discrete-time system

$$x(k+1)=\begin{bmatrix} 1.0078 & 0.0301 \\ 0.5202 & 1.0078 \end{bmatrix} x(k)+\begin{bmatrix} -0.0001 \\ -0.0053 \end{bmatrix} u(k). \tag{61}$$

Then, the poles of the system (61) are 1.1329 and 0.8827, thus this system is unstable. In Table 3, the obtained delay bounds when $h_m=1$ and the corresponding controller by applying Theorem 3 are compared with those of [9] and [10]. From Table 3, with smaller controller gain than those of [9] and [10], Theorem 3 gives larger delay bounds. To confirm this result, simulation result by applying our derived controller gain is included in Fig. 2 which shows that the state responses converge to zero when initial values of the states are $x(0)=[1,-1]^T$ and time-delay $h(k)$ is assumed as $h(k)=6|\sin(\pi k/2)|+1\in[1,7]$.

**Example 3.** Consider another practical system which is the satellite system [28,29]. The satellite system shown in Fig. 3 [29] has two rigid bodies joined by a flexible link. The

Table 1
Maximum bounds $h_M$ with different $h_m$ (Example 1).

| Method | 0 | 2 | 4 | 6 | 7 | 10 | 12 | 13 | 15 | 16 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| [8] | 6 | 7 | 8 | 9 | 10 | 12 | 13 | 14 | 16 | 17 | 20 | 25 | 30 |
| [9] | 12 | 13 | 13 | 14 | 14 | 15 | 17 | 17 | 18 | 19 | 22 | 26 | 30 |
| [10] | 12 | 13 | 13 | 14 | 15 | 17 | 18 | 19 | 20 | 21 | 24 | 29 | 33 |
| [11] | 12 | 14 | 15 | 16 | 16 | 18 | 20 | 20 | 21 | 23 | 25 | 30 | 34 |
| [13] | 17 | 17 | 17 | 18 | 18 | 20 | 21 | 22 | 23 | 24 | 27 | 31 | 35 |
| [14] | 17 | 17 | 17 | 18 | 18 | 20 | 21 | 22 | 23 | 24 | 27 | 31 | 35 |
| Theorem 1 | 16 | 17 | 17 | 18 | 19 | 20 | 21 | 22 | 24 | 24 | 27 | 31 | 36 |
| Theorem 2 | 22 | 22 | 22 | 22 | 22 | 23 | 23 | 24 | 25 | 26 | 28 | 32 | 36 |

Table 2
Maximum bounds $h_M$ when $h_m = 16$ (Example 1).

| Method | $h_M$ | Number of variables |
|---|---|---|
| [9] | 19 | 143 |
| [12] ($m = 1$, $\tau = 16$) | 24 | 138 |
| [12] ($m = 2$, $\tau = 8$) | 24 | 195 |
| [12] ($m = 4$, $\tau = 4$) | 25 | 345 |
| [12] ($m = 8$, $\tau = 2$) | 25 | 789 |
| Theorem 1 | 24 | 117 |
| Theorem 2 | 26 | 126 |

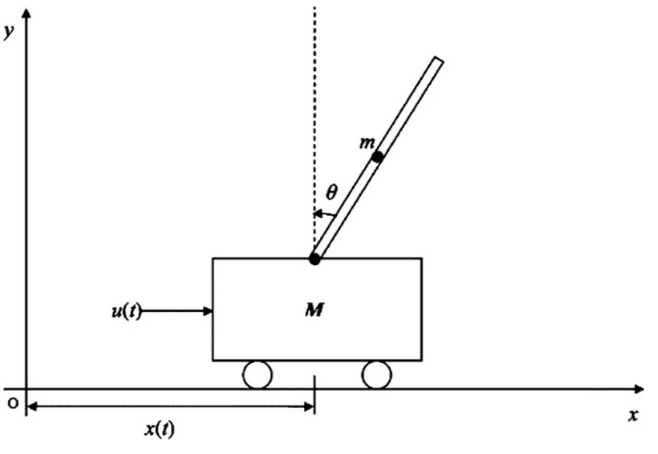


Fig. 1. Inverted pendulum system (Example 2).

Table 3
Maximum bounds $h_M$ and controller gains $K$ when $h_m = 1$ (Example 2).

| Method | $h_M$ | $K$ |
|---|---|---|
| [9] | 3 | [102.9100, 80.7916] |
| [10] | 4 | [110.6827, 34.6980] |
| Theorem 3 ($\delta = 1000$) | 7 | [98.5858, 24.0621] |

dynamic equations of this system are as follows:

$$J_1\ddot{\theta}_1(t) + f(\dot{\theta}_1(t) - \dot{\theta}_2(t)) + k(\theta_1(t) - \theta_2(t)) = u(t),$$
$$J_2\ddot{\theta}_2(t) + f(\dot{\theta}_1(t) - \dot{\theta}_2(t)) + k(\theta_1(t) - \theta_2(t)) = 0, \tag{62}$$

where $J_i (i = 1,2)$ are the moments of inertia of the two bodies (the main body and the instrumentation module), $f$ is a viscous damping, $k$ is a torque constant, $\theta_i(t)(i = 1,2)$ are the yaw angles for the two bodies, and $u(t)$ is a control input. Assume $J_i(i = 1,2) = 1$, $k = 0.09$, $f = 0.04$, and state vector $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]^T = [\theta_1(t), \theta_2(t), \dot{\theta}_1(t), \dot{\theta}_2(t)]^T$. By choosing sampling time $T_s = 10$ ms, the system (62) can be transformed as the following discrete

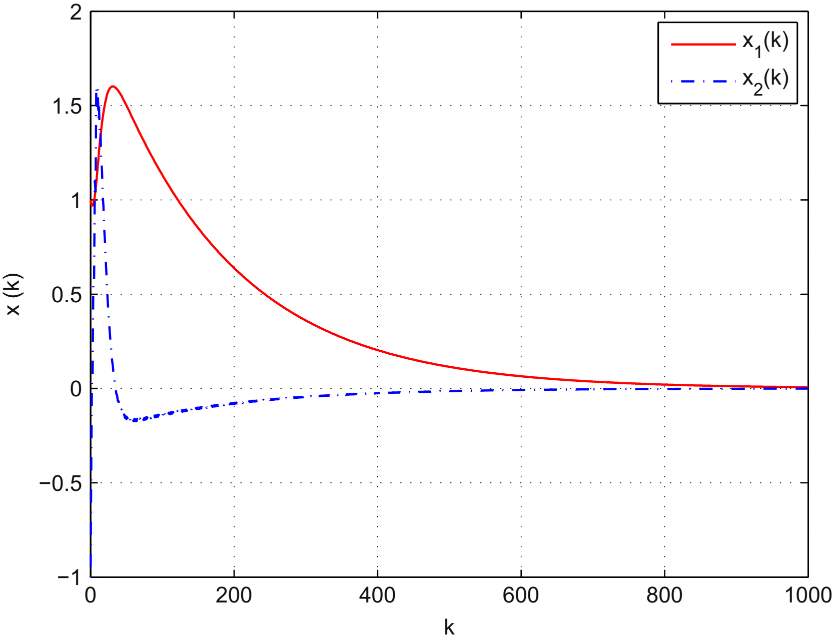


Fig. 2. State responses of inverted pendulum system (Example 2).

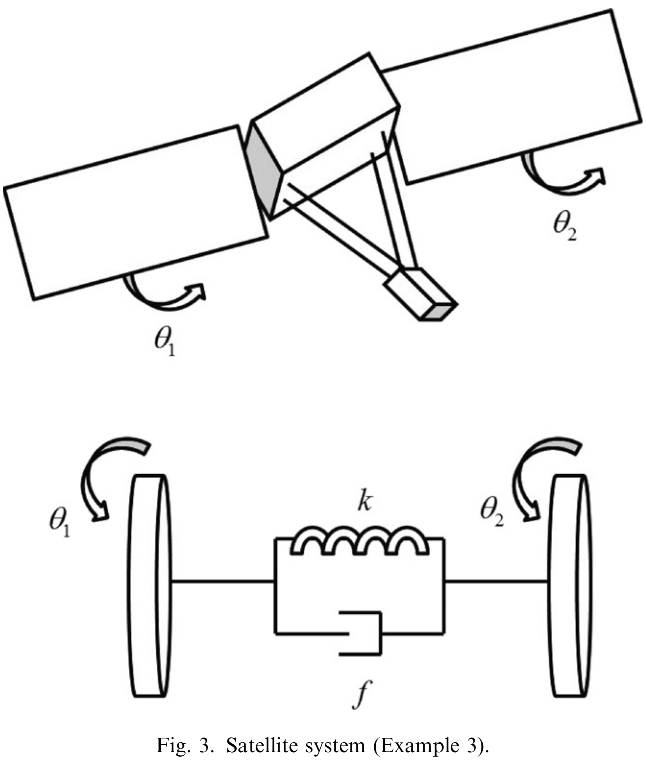


Fig. 3. Satellite system (Example 3).

system:

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0.01 & 0 \\ 0 & 1 & 0 & 0.01 \\ -0.009 & 0.009 & 0.9996 & 0.0004 \\ 0.009 & -0.009 & 0.0004 & 0.9996 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 0.01 \\ 0 \end{bmatrix} u(k). \tag{63}$$

By applying Theorem 3 to the above system when $h_m = 1$ and $\delta = 110$, the maximum delay bound can be obtained as $h_M = 109$ and the corresponding time-delayed controller gain as $K = [0.1284, -0.1380, -0.3049, 0.0522]$. With the controller gain obtained by Theorem 3, one can get maximum delay bounds listed in Table 4. From Table 4, our proposed stability criteria significantly enhance the feasible region comparing with those of [8–11] and [13,14]. To

Table 4
Maximum bounds $h_M$ when $h_m = 1$ (Example 3).

| Method | $h_M$ |
|---|---|
| [8] | 40 |
| [11] | 71 |
| [9] | 98 |
| [10] | 98 |
| [13] | 98 |
| [14] | 98 |
| Theorem 1 | 129 |
| Theorem 2 | 135 |

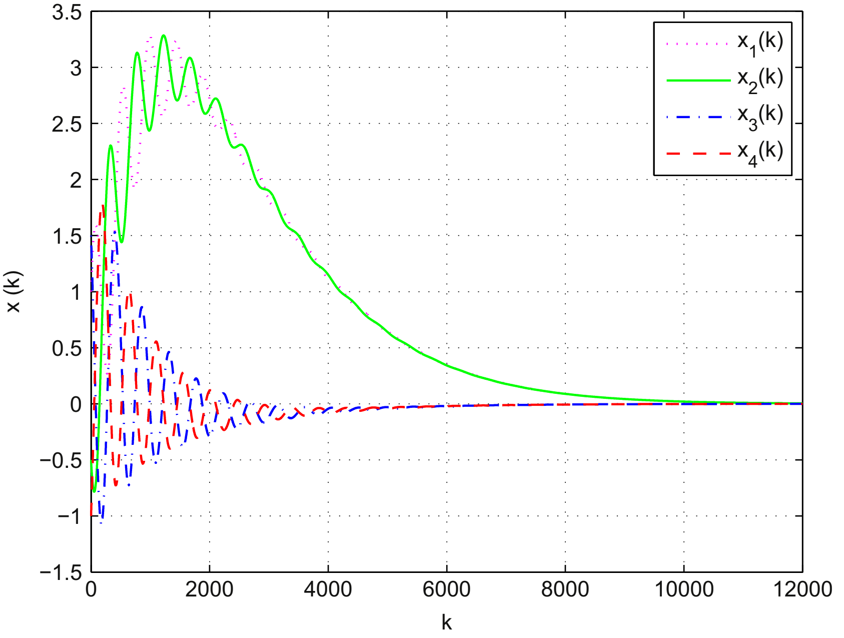


Fig. 4. Responses of the satellite system (Example 3).

check the effectiveness of the obtained results, simulation result by applying our derived controller gain is included in Fig. 4 which shows that the state responses converge to zero when initial values of the states are $x(0)=[1,-0.5,1.5,-1]^T$ and time-delay $h(k)$ is assumed as $h(k)=134|\sin(\pi k/2)|+1\in[1,135]$.

## 5. Conclusions

In this paper, two delay-dependent stability criteria and one stabilization criterion for the discrete-time systems with interval time-varying delays have been proposed. By constructing the suitable augmented Lyapunov–Krasovskii functional and vector, the improved stability criterion was derived in Theorem 1. Based on the results of Theorem 1 and utilizing zero equalities (44)–(46), the further improved result was proposed in Theorem 2. Also, the designing method of time-delayed controller gain was presented in Theorem 3. Three numerical examples have been given to show the effectiveness and usefulness of the presented criteria.

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