Design of State Estimator for Discrete-Time Neural Networks: Delay-Independent Approach

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In this Letter, the design problem of state estimator for a class of discrete-time neural networks is studied. A delay-independent linear matrix inequality (LMI) criterion for existence of the estimator is derived by using the Lyapunov method. The criterion can be easily solved by various convex optimization algorithms. A numerical example with simulation results is given to show the effectiveness of proposed method.

KEYWORDS: discrete-time neural networks, state estimation, LMI

The last few decades have witnessed a large amount of successful applications of neural networks in various areas including signal processing, pattern recognition, fixed-point computation, combinatorial optimization and associative memories.1–7 It should be pointed out that most of neural networks discussed in the literature have been assumed to act in a continuous-time manner. However, when implementing these networks for practical uses, discrete-time models should be formulated. As pointed out in,8) the discretization of continuous-time model cannot preserve the dynamics of the continuous-time counterpart even for a small sampling period. Hence there is a strong need to investigate the dynamics of discrete-time neural networks. In this regard, the dynamics analysis problem for discrete-time neural networks with or without time delays has received considerable interest in recent years.9,10)

On the other hand, since the neuron states are not often fully available in the network outputs in some applications, the state estimation problem of neural networks becomes an important topic for real applications recently.11,12) As is well known, time delay may occur in the process of information storage and transmission in neural networks and its existence is often a source of oscillation and instability. This leads to the model of delayed cellular neural networks. Recently, Wang et al.11) and Huang et al.12) derived delay-independent and delay-dependent criteria for state estimator design of delayed neural networks, respectively. However, up to date, the state estimation problem for discrete-time neural networks with time delays has not fully investigated. Thus, the main objective of this work is to estimate the neuron states of the network through available output measurements such that the dynamics of the closed-loop error system is globally stable. By constructing a suitable Lyapunov functional and utilizing linear matrix inequality (LMI) framework, a novel delay-independent criterion for the existence of proposed state estimator of the network is given in terms of LMI. The advantage of the proposed approach is that resulting criterion can be solved efficiently via existing numerical convex optimization algorithms.13) Throughout the Letter, I denotes the identity matrix with appropriate dimensions. The superscript “T” represents the transpose of given vector or matrix. ∥·∥ denotes the Euclidean norm of given vector. • denotes the elements below the main diagonal of a symmetric block matrix. For symmetric matrices X and Y, the notation X > Y (respectively, X ≥ Y) means that the matrix X − Y is positive definite (respectively, nonnegative).

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diag{⋯} denotes the block diagonal matrix.

In this Letter, the following discrete-time neural networks with delays is considered:

\[ x(k + 1) = Ax(k) + B_1 f(x(k)) + B_2 f(x(k - d)) + J(k), \]
\[ y(k) = Cx(k) + z(t, x(k)) \]

where \( x(k) = [x_1(k), \ldots, x_n(k)]^T \in \mathbb{R}^n \) is the neuron state vector associated with \( n \) neurons, \( f(x(k)) = [f_1(x_1(k)), \ldots, f_n(x_n(k))]^T \in \mathbb{R}^n \) is the neuron activation functions, \( J(k) = [J_1(k), J_2(k), \ldots, J_n(k)]^T \) is the external input vector at time \( k \), \( y(k) \in \mathbb{R}^l \) is the measurement output, \( d \) is a positive integer and corresponds to finite speed of axonal signal transmission delay, \( A = \text{diag}[a_1, a_2, \ldots, a_n] \) is a positive diagonal matrix with \( 0 < a_i < 1 \) \( (i = 1, \ldots, n) \), \( B_1 \) and \( B_2 \) are known matrices and the interconnection matrices representing the weight coefficients of the neurons, \( C \in \mathbb{R}^{l \times n} \) is a known output weighting matrix, and \( z : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^l \) is the neuron-dependent nonlinear disturbances on the network outputs.

Throughout the Letter, it is assumed that the functions \( f \) and \( z \) are Lipschitz continuous:

\[ |f(x_1) - f(x_2)| \leq |F(x_1 - x_2)|, \]
\[ |z(x_1) - z(x_2)| \leq |Z(x_1 - x_2)|, \]

where \( F \in \mathbb{R}^{n \times n} \) and \( Z \in \mathbb{R}^{n \times n} \) is the known constant matrices.

The purpose of this Letter is to present an efficient estimation algorithm to observe the neuron states from the available network output. For this end, the following full-order observer is proposed:

\[ \hat{x}(k + 1) = A\hat{x}(k) + B_1 f(\hat{x}(k)) + B_2 f(\hat{x}(k - d)) + J(k) + K(y(k) - C\hat{x}(k) - z(t, \hat{x}(k))), \]

where \( \hat{x}(k) \in \mathbb{R}^n \) is the estimation of the neuron state, and \( K \in \mathbb{R}^{n \times n} \) is the gain matrix of the estimator to be designed later.
Define the error state to be \( e(k) = x(k) - \bar{x}(k) \). Then, the error dynamical system is expressed by

\[
e(k+1) = (A - KC)e(k) + B_1\phi(k) + B_2\phi(k-d) - K\psi(k),
\]

where \( \phi(k) = f(x(k)) - f(\bar{x}(k)) \) and \( \psi(k) = z(k, x(k)) - z(k, \bar{x}(k)) \).

The following fact will be used for deriving main result.

**Fact 1. (Schur complement)** Given constant symmetric matrices \( \Sigma_1, \Sigma_2, \Sigma_3 \) where \( \Sigma_1 = \Sigma_1^T \) and \( 0 < \Sigma_2 = \Sigma_2^T \), then
\[
\begin{bmatrix}
\Sigma_1 & \Sigma_3^T \\
\Sigma_3 & -\Sigma_2
\end{bmatrix} < 0, \quad \text{or} \quad
\begin{bmatrix}
-\Sigma_2 & \Sigma_3 \\
\Sigma_3^T & -\Sigma_1
\end{bmatrix} < 0.
\]

Now, we derive a new delay-independent criterion for design of state estimator (3) using the Lyapunov method combining with LMI framework.

Then we have the following theorem.

**Theorem 1.** For given matrices \( F, Z \), the error system (4) is globally asymptotically stable if there exist positive definite matrices \( P, Q \), positive scalars \( \alpha_i \) (\( i = 1, 2, 3 \)) and any matrix \( Y \) satisfying the following LMI:

\[
\begin{bmatrix}
-P & PA - YC & 0 & PB_1 & PB_2 & -Y \\
* & \Gamma & 0 & 0 & 0 & 0 \\
* & * & -Q + \alpha_2F^TF & 0 & 0 & 0 \\
* & * & * & -\alpha_1I & 0 & 0 \\
* & * & * & * & -\alpha_2I & 0 \\
* & * & * & * & * & -\alpha_3I
\end{bmatrix} < 0,
\]

where \( \Gamma = -P + Q + \alpha_1F^TF + \alpha_3Z^TZ \).

Then, the gain matrix \( K \) of the state estimator (3) is given by \( K = P^{-1}Y \).

**Proof.** Let us consider the Lyapunov functional candidate:

\[
V = e^T(k)Pe(k) + \sum_{i=k-d}^{k-1} e^T(i)Qe(i).
\]

Then, we have

\[
\Delta V = [(A - KC)e(k) + B_1\phi(k) + B_2\phi(k-d) - K\psi(k)]^T P(A - KC)e(k) + B_1\phi(k) + B_2\phi(k-d) - K\psi(k)]
\]

\[
- e^T(k)Pe(k) + e^T(k)Qe(k) - e^T(k-d)Qe(k-d).
\]

From (2) and definitions of \( \phi \) and \( \psi \), it is clear that

\[
\phi^T(k)\phi(k) = |f(x(k)) - f(\bar{x}(k))|^2 \leq |Fe(k)|^2 = e^T(k)F^TFe(k),
\]

\[
\phi^T(k-d)\phi(k-d) \leq |Fe(k-d)|^2 = e^T(k-d)F^TFe(k-d),
\]

\[
\psi^T(k)\psi(k) \leq |Ze(k)|^2 = e^T(k)Z^TZe(k).
\]

Then, for positive scalars \( \alpha_i \) (\( i = 1, 2, 3 \)), we have

\[
\alpha_1[e^T(k)F^TFe(k) - \phi^T(k)\phi(k)] \geq 0,
\]

\[
\alpha_2[e^T(k-d)F^TFe(k-d) - \phi^T(k-d)\phi(k-d)] \geq 0,
\]

\[
\alpha_3[e^T(k)Z^TZe(k) - \psi^T(k)\psi(k)] \geq 0.
\]

Utilizing the relationship (8)–(9), thus we have the following a new bound for \( \Delta V \):

\[
\Delta V \leq \xi^T(k)\Sigma\xi(k)
\]

where

\[
\Sigma = \begin{bmatrix}
(1,1) & 0 & (A - KC)^T PB_1 & (A - KC)^T PB_2 & -(A - KC)^T PK \\
* & -Q + \alpha_2F^TF & 0 & 0 & 0 \\
* & * & -\alpha_1I + B_1^T PB_1 & B_1^T PB_2 & -B_1^T PK \\
* & * & * & B_2^T PB_2 - \alpha_2I & -B_2^T PK \\
* & * & * & * & K^T PK - \alpha_3I
\end{bmatrix}.
\]

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with \((1, 1) = (A - KC)^TP(A - KC) - P + Q + \alpha_1F^TF + \alpha_2Z^TZ\) and \(\zeta = [e^T(k) \, e^T(k - d) \, \phi^T(k) \, \phi^T(k - d) \, \psi^T(k)]\). If the matrix \(\Sigma\) is a negative definite matrix, then \(\Delta V < 0\), which guarantees the stability of the error system. By Fact 1, the inequality, \(\Sigma < 0\), is equivalent to the following inequality:

\[
\begin{bmatrix}
-P^{-1} & A - KC & 0 & B_1 & B_2 & -K \\
* & \Gamma & 0 & 0 & 0 & 0 \\
* & * & -Q + \alpha_2F^TF & 0 & 0 & 0 \\
* & * & * & -\alpha_1I & 0 & 0 \\
* & * & * & * & -\alpha_2I & 0 \\
* & * & * & * & * & -\alpha_3I
\end{bmatrix} < 0.
\] (12)

Postmultiplying and premultiplying the matrix inequality (12) by the matrix \(\text{diag}\{P, I, I, I, I, I\}\) and by its transpose, respectively, gives

\[
\begin{bmatrix}
-P & P(A - KC) & 0 & PB_1 & PB_2 & -PK \\
* & \Gamma & 0 & 0 & 0 & 0 \\
* & * & -Q + \alpha_2F^TF & 0 & 0 & 0 \\
* & * & * & -\alpha_1I & 0 & 0 \\
* & * & * & * & -\alpha_2I & 0 \\
* & * & * & * & * & -\alpha_3I
\end{bmatrix} < 0.
\] (13)

By defining \(Y = PK\), the inequality (13) is equivalent to LMI (5) by Fact 1. Therefore, if the LMI condition (5) holds, this implies that the error dynamics (4) is asymptotically stable by the Lyapunov theory. This completes our proof. □

**Remark 1.** The solutions of Theorem 1 can be obtained by solving the eigenvalue problem with respect to solution variables, \(P, Y, Q\), and \(\alpha_i\) \((i = 1, 2, 3)\), which is a convex optimization problem.\(^{13}\) In this Letter, we utilize Matlab’s LMI Control Toolbox\(^{14}\) which implements interior-point algorithm. This algorithm is significantly faster than classical convex optimization algorithms.\(^{13}\)

**Remark 2.** Our criterion given in Theorem 1 is delay-independent. In general, when the delay is large, the delay-independent criterion is less conservative than delay-dependent criterion.

In the following, a numerical example with simulation results is provided to demonstrate the effectiveness of the proposed method. Consider the delayed neural network with the following parameters:

\[
A = \text{diag}\{0.3, 0.5, 0.4\},
\]

\[
B_1 = \begin{bmatrix}
0.2 & -0.1 & 0 \\
0.1 & 0.3 & -0.2 \\
-0.2 & 0.1 & 0.2
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0.1 & 1 & 0.2 \\
-0.1 & 0.2 & 0 \\
-0.2 & 0.1 & 0.2
\end{bmatrix}, \quad J = \begin{bmatrix}
-\sin(k\pi/4) \\
-\sin(k\pi/4) \\
2\cos(k\pi/4)
\end{bmatrix},
\]

and \(C = I, f(x) = 0.5 \sin(x(k)), z(x(k)) = 0.2 \sin(4x(k)), x(0) = [1 \, -1 \, 0.5]^T\). From the functions \(f, z\), it is easy to see that \(F = 0.5I, Z = 0.2I\).

Now, by applying Theorem 1 to above system, we can see that the LMI given in Theorem 1 is feasible and a solution set is obtained as follows:

\[
P = \begin{bmatrix}
88.8415 & -12.3946 & 3.3245 \\
* & 142.7542 & 4.9865 \\
* & * & 138.8337
\end{bmatrix}, \quad Y = \begin{bmatrix}
22.1393 & -3.3583 & 0.7789 \\
-0.6167 & 48.7140 & 0.4584 \\
0.4325 & 0.9774 & 38.8970
\end{bmatrix},
\]

\[
Q = \begin{bmatrix}
33.1149 & -6.3718 & 1.7753 \\
* & 59.7249 & 2.4915 \\
* & * & 58.9004
\end{bmatrix}, \quad \alpha_1 = 0.5, \quad \alpha_2 = 0.05, \quad \alpha_3 = 0.005.
\]

From the solutions \(P\) and \(Y\), then the gain matrix, \(K\), of state estimator is

\[
K = \begin{bmatrix}
0.2518 & 0.0102 & -0.0027 \\
0.0177 & 0.3423 & -0.0068 \\
-0.0035 & -0.0055 & 0.2805
\end{bmatrix}.
\]

The simulation result with the obtained estimator gain \(K\) and time delay \(d = 5\) is shown in Fig. 1. From the figure, we can see that the responses of the state estimators track to true states very well.
Fig. 1. The true state $x_i(k)$ and its estimate $\hat{x}_i(k)$. 

12) H. Huang, G. Feng, and J. Cao: to be published in Neurocomputing.