A New Approach to Synchronization of Discrete-Time Chaotic Systems

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(Received June 24, 2007; accepted July 30, 2007; published September 10, 2007)

This letter deals with the synchronization of discrete-time chaotic systems. Based on the Lyapunov method, we present a new control method that guarantees that the synchronized states are asymptotically stable. The controller consists of two parts: one is a linear dynamic controller, and the other is a nonlinear control law for compensating the nonlinear terms of the error signal. Then, the existence criterion of the controller for the synchronization is derived. The criterion is expressed as a linear matrix inequality (LMI), readily solvable by various convex optimization algorithms.

KEYWORDS: discrete-time, chaotic systems, synchronization, LMI, Lyapunov method
DOI: 10.1143/JPSJ.76.093002

Chaos is a very interesting nonlinear phenomenon and has applications in many areas such as biology, economics, signal generator design, secure communication, and many other engineering systems. Chaotic synchronization is one of the important and hot issues in the research field of chaos systems.1–12) During the last decade, the chaotic synchronization of continuous-time systems has been extensively studied and many methods such as identical synchronization, phase synchronization, lag synchronization, anticipated synchronization, generalized synchronization, and projected synchronization have been developed. Further, from the beginning with the remarkable works,13,14) the synchronization problem of discrete-time chaotic systems has been investigated by some researchers.15,16) However, there is room for further development in comparison with studies in the continuous-time domain. This research is motivated in part by the potential applications of discrete-time chaotic systems in several fields such as secure digital communication.

This letter considers the chaos synchronization of discrete-time chaotic systems. Motivated by a recent work,11) a new feedback controller, which consists of a linear dynamic feedback controller and a nonlinear active feedback one, is designed. In some real control situations, there is a strong need to construct a dynamic feedback controller instead of a static feedback controller in order to obtain a better performance and dynamical behavior of the state response. The dynamic controller will provide more flexibility compared to the static controller and the apparent advantage of this type of controller is that it provides more free parameters for selection. Further, it is well known that the dynamic controller is superior to the static one in terms of the ability to cope with system uncertainty and disturbance.17,18)

For stability analysis, the Lyapunov stability theory and linear matrix inequality (LMI) framework are utilized in this work. Then, the controller existence criterion for the synchronization of the systems is derived in terms of LMIs, which can be easily solved by various convex optimization algorithms developed recently.20)

Throughout the letter, \( \mathcal{R}^n \) denotes \( n \)-dimensional Euclidean space, and \( \mathcal{R}^{n \times m} \) is the set of all \( n \times m \) real matrices. \( X < 0 \) means that \( X \) is a real symmetric negative definite matrix. \( I \) denotes the identity matrix with appropriate dimensions. \( \| \cdot \| \) refers to the Euclidean vector norm or the induced matrix 2-norm.

Consider a class of discrete-time chaotic systems described by

\[
\begin{align*}
x(k + 1) &= Ax(k) + f(x(k)), \quad (1) \\
y(k + 1) &= Ay(k) + f(y(k)) + Hu(k) + \alpha(k), \quad (2)
\end{align*}
\]

where \( x \in \mathcal{R}^n \) is the state vector, \( A \in \mathcal{R}^{n \times n} \) is the system matrix representing the linear part of the dynamics, and \( f(x) : \mathcal{R}^n \rightarrow \mathcal{R}^n \) is the nonlinear part of system (1). Equation (1) is considered as the drive system, and the response system with control inputs is introduced as follows:

\[
y(k + 1) = Ay(k) + f(y(k)) + Hu(k) + \alpha(k),
\]

where \( y \in \mathcal{R}^n \) is the state vector, \( f(x) : \mathcal{R}^n \rightarrow \mathcal{R}^n \) is a continuous nonlinear vector function, \( H \in \mathcal{R}^{n \times p} \) is the input matrix, and \( u \in \mathcal{R}^p \) and \( \alpha \in \mathcal{R}^n \) are the linear dynamic feedback control law and nonlinear static feedback one, respectively.

The objective of chaos synchronization is how to design the controllers, which can synchronize the states of both the drive and response systems. If we define the error vector as \( e(k) = y(k) - x(k) \), the dynamic equation of the synchronization error can be expressed as

\[
e(k + 1) = Ae + \tilde{f} + Hu + \alpha,
\]

where \( \tilde{f} = f(y(k)) - f(x(k)) \). Hence, the objective of synchronization is to make \( \lim_{k \to \infty} \| e(k) \| = 0 \). In order to synchronize two chaotic systems, we first design the control input \( \alpha \) to compensate the nonlinear terms:

\[
\alpha = -\tilde{f}.
\]

Note that this kind of compensation is widely used in the active control method for chaotic systems. Then, the controlled error system is simplified as

\[
e(k + 1) = Ae(k) + Hu(k).
\]

Now, let us design the dynamic control law \( u \) for the stability of the error system. For this purpose, we propose the following linear dynamic controller:

\[
\begin{align*}
\xi(k + 1) &= A_e \xi(k) + B_e e(k), \\
u(k) &= C_e \xi(k), \quad \xi(0) = 0,
\end{align*}
\]

where \( \xi(k) \in \mathcal{R}^n \) is the controller state, and \( A_e, B_e, \) and \( C_e \) are gain matrices with appropriate dimensions to be determined later.
By applying this controller (4)–(6) to system (5), we obtain the closed-loop system
\[ z(k + 1) = \hat{A}z(k) \]
where
\[ z(k) = \begin{bmatrix} e(k) \\ \xi(k) \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A & HC_c \\ B_c & A_c \end{bmatrix}. \]

Then, we have the following main result.

**Theorem 1.** There exists a dynamic controller (6) for the error system (3) if there exist positive definite matrices \( S, Y \), and matrices \( \hat{A}, \hat{B}, \hat{C} \) satisfying the following LMIs:
\[
\begin{bmatrix}
-Y & -I & YA^T & \hat{C}^T H^T & \hat{A}^T \\
* & -S & A^T & A^T S + \hat{B}^T \\
* & * & -Y & -I \\
* & * & * & -S \\
\end{bmatrix} < 0.
\]

Then, the error system (3) is asymptotically stable via the control laws (4) and (6). That is, the synchronization between the drive system (1) and the response system (2) is accomplished.

**Proof.** First, consider a positive definite matrix satisfying the following relationship:
\[
P = \begin{bmatrix} S & N \\ N^T & U \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y & M \\ M^T & W \end{bmatrix}, \quad MN^T = I - YS.
\]

For the simplification of the representation given in eq. (15), let us define a new set of variables as follows:
\[
\hat{A} = SAY + SH\hat{C} + \hat{B}Y + N_A M^T, \\
\hat{B} = NB_c, \quad \hat{C} = C_c M^T.
\]

Thus, the inequality (15) is simplified to the LMI (8). For the positiveness of \( P \), the following condition is needed from its definition:
\[
\begin{bmatrix} Y & I \\ I & S \end{bmatrix} > 0,
\]
in which the condition (17) holds if the LMI (8) is satisfied. This completes the proof. \( \square \)

**Remark 1.** Given any solution of the LMI (8) in Theorem 1, a corresponding controller of the form (6) will be constructed as follows:
- Compute the invertible matrices \( M \) and \( N \) satisfying \( MN^T = I - YS \) using matrix algebra.
- Utilizing the matrices \( M \) and \( N \) obtained above, solve the following system of equations:
\[
\hat{B} = NB_c,
\]
where \( S, Y \in \mathbb{R}^{n \times n} \) are positive definite matrices, and \( M \) and \( N \) are invertible matrices. Next, let us take the following legitimate Lyapunov functional candidate:
\[
V(k) = z^T(k)Pz(k).
\]

The corresponding Lyapunov difference is
\[
\Delta V = V(k + 1) - V(k) = z^T(k)(A^T P A - P)z(k).
\]
Therefore, if \( A^T P A - P < 0 \), there exist positive scalars \( \gamma \) such that \( \Delta V < -\gamma \| z(k) \|^2 < 0 \), which guarantees the asymptotic stability of the system by the Lyapunov stability theory. By Schur complements, the inequality \( A^T P A - P < 0 \) is equivalent to the LMI:
\[
\begin{bmatrix} -P & A^T P \\ * & -P \end{bmatrix} < 0.
\]

Define
\[
F_1 = \begin{bmatrix} Y & I \\ M^T & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} I & S \\ 0 & N^T \end{bmatrix}.
\]

Then, it follows that
\[
P F_1 = F_2, \quad F_1^T P F_1 = F_1^T F_2 = \begin{bmatrix} Y & I \\ I & S \end{bmatrix} > 0.
\]

Now, by postmultiplying and premultiplying the inequality (11) by the matrix \( F_1 \) and its transpose, respectively, we obtain
\[
- F_1^T F_2 = F_1^T \hat{A}^T \hat{A} F_2 < 0.
\]

From the relations (9), (12)–(13), it is obvious that eq. (14) is equivalent to
\[
\begin{bmatrix} -Y & -I & YA^T & MC_c^T H^T & YA^T S + MC_c^T H^T S + YB_c^T N^T + M_A^T N^T \\ * & S & A^T \\ * & * & -Y \\ * & * & * \end{bmatrix} < 0.
\]

For the Hénon mapping to demonstrate the effectiveness of the proposed method. Consider the following Hénon mapping as the drive system:
\[
\begin{align*}
x_1(k + 1) &= x_2(k) + 1 - ax_1^2(k), \\
x_2(k + 1) &= b x_1(k).
\end{align*}
\]

It is well known that the Hénon map has a strange attractor when \( a = 1.4 \) and \( b = 0.3 \). The drive system (19) can be written in the form of
Now, let us consider the control problem for synchronization. First, the nonlinear control law \( \alpha(k) \) for compensating the nonlinear terms of the system is

\[
\alpha(k) = -\left[ -1.4y_1^2(k) + 1.4x_1^2(k) \right].
\]

Next, let us design the dynamic controller (6). By solving the problem given in Theorem 1 using the LMI Control Toolbox, we have a possible stabilizing dynamic feedback controller for the system (21):

\[
A_c = \begin{bmatrix} 0 & 0 \\ -0.3874 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1.0001 \end{bmatrix}, \quad C_c = \begin{bmatrix} -0.3000 & 0 \end{bmatrix}.
\]

Then, by the control inputs (22) and the dynamic controller (23), the simulation result is illustrated in Fig. 2, in which the control inputs are applied at \( k = 40 \). We see that the synchronization error rapidly approaches zero.