SYNCHRONIZATION OF NEURAL NETWORKS OF NEUTRAL TYPE WITH STOCHASTIC PERTURBATION

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In this letter, the problem of feedback controller design to achieve synchronization for neural network of neutral type with stochastic perturbation is considered. Based on Lyapunov method and LMI (linear matrix inequality) framework, the goal of this letter is to derive an existence criterion of the controller for the synchronization between master and response networks.

Keywords: Neural networks; stochastic perturbation; LMI; Lyapunov method.

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1. Introduction

During the last two decades, neural networks have been applied to various engineering problems such as image processing, associative content-addressable memories, fixed point computation, and pattern classification. Since significant time delays are ubiquitous both in neural processing and in signal transmission it is necessary to introduce delays into communication channels which lead to delayed neural networks (DCNNs) model. Thus the stability analysis of DCNNs has become an important topic of theoretical studies in neural networks. Recently, it was revealed that if the neural network’s parameters and time delays are appropriately chosen, the DCNNs can exhibit some complicated dynamics and even chaotic behaviors. Hence, it has attracted many scholars to study the synchronization of chaotic DCNNs.
In real nervous systems, synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters. Therefore, it is of practical importance to study the stochastic effects on the stability property of delayed neural networks.

In Li and Cao, the synchronization problem for DCNNs with stochastic perturbation has been investigated. On the other hand, due to the complicated dynamic properties of the neural cells in the real world, the existing neural network models in many cases cannot characterize the properties of a neural reaction process precisely. It is natural and important that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions. For example, in very large scale integration (VLSI) implementations of artificial neural networks, delay transmission line and partial element equivalent circuit (PEEC) are two main kinds of elements to produce delays. Note that the circuits established by PEEC may cause neutral delays.

This letter considers neural networks of neutral type with stochastic perturbation. To date, the synchronization problem for neural networks of neutral type with stochastic perturbation has not been investigated fully. In this letter, the state feedback control scheme is proposed in order to achieve the synchronization between master and response networks. By constructing suitable Lyapunov functional, a delay-dependent criterion, which is less conservative than a delay-independent one when the size of delays is small, for the existence of the controller is presented. The criterion is represented in terms of LMI. A numerical example is illustrated to show the effectiveness of the proposed method.

Notation. $\mathcal{R}^n$ denotes the $n$-dimensional Euclidean space, and $\mathcal{R}^{m\times n}$ is the set of $m \times n$ real matrix. $|\cdot|$ represents the absolute value. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices $X$ and $Y$, the notation $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite (respectively, nonnegative). $\text{diag}\{\cdots\}$ denotes the block diagonal matrix. $*$ represents the elements below the main diagonal of a symmetric matrix. $I$ is the identity matrix with appropriate dimension. The notation $\rho(A)$ denotes the spectral radius of $A$. $\lambda_{\max}(A)$ and $\text{trace}(A)$ represent the maximum eigenvalue and trace of matrix $A$, respectively. For $h > 0$, $\mathcal{C}([-h, 0], \mathcal{R}^n)$ means the family of continuous functions $\phi$ from $[-h, 0]$ to $\mathcal{R}^n$ with the norm $\|\phi\| = \sup_{-h \leq s \leq 0} |\phi(s)|$. Let $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $F_0$ contains all $\mathcal{P}$-null sets). $L^p_{\mathcal{F}_0}([-h, 0], \mathcal{R}^n)$ the family of all $\mathcal{F}_0$-measurable $\mathcal{C}([-h, 0], \mathcal{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$, where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$. Denote by $\mathcal{C}^{2,1}(\mathcal{R}^n \times \mathcal{R}^+, \mathcal{R}^+)$ the family of all nonnegative functions $V(x, t)$ on $\mathcal{R}^n \times \mathcal{R}^+$ which are continuously twice differentiable in $x$ and differentiable in $t$. 
2. Problem Statements

Consider a class of neural networks with time-varying delays

\[
d[x(t) - Cx(t - h(t))] = [-Ax(t) + W_0 f(x(t)) + W_1 f(x(t - h(t))) + J]dt,
\]

where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \) is the neuron state vector, \( n \) denotes the number of neurons in a neural network, \( f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T \in \mathbb{R}^n \) denotes the neuron activation function, \( A = \text{diag}(a_i) \) is a positive diagonal matrix, \( W_0 = (w_{ij}^0)_{n \times n}, W_1 = (w_{ij}^1)_{n \times n}, \) and \( C = (c_{ij})_{n \times n} \) are the interconnection matrices representing the weight coefficients of the neurons, \( J = [J_1, J_2, \ldots, J_n]^T \) means a constant input vector, and \( h(t) \) is time-varying delay. In this letter, it is assumed that \( 0 \leq h(t) \leq \bar{h} \) and \( \dot{h}(t) \leq h_d < 1 \), and the matrix \( C \) satisfies \( \rho(C) < 1 \).

The activation functions, \( f_i(x_i(t)), i = 1, 2, \ldots, n, \) are assumed to be non-decreasing, bounded and globally Lipschitz; that is,

\[
|f_j(x_j)| \leq |l_j(x_j)|, \quad \forall x_j \neq 0, \quad j = 1, \ldots, n,
\]

where \( l_i \) is a known constant.

For synchronization problem, let us take system (1) as a drive network, then, we construct the response networks as follows:

\[
d[y(t) - Cy(t - h(t))] = [-Ay(t) + W_0 f(y(t)) + W_1 f(y(t - h(t))) + J + u(t)]dt
\]

\[
+ [\sigma(t, e(t), e(t - h(t))]d\omega(t),
\]

where \( y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in \mathbb{R}^n \) is the state vector of response network, \( u(t) \) is the control input for achieving synchronization, \( \omega(t) \) is a scalar Wiener process (Brownian motion) on \((\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P})\) which satisfies \( \mathbb{E}\{d\omega(t)\} = 0 \) and \( \mathbb{E}\{d\omega^2(t)\} = dt, \) and \( \sigma(\cdot) \) is the noise intensity matrix. This type of stochastic perturbation can be regarded as a result from the occurrence of external random fluctuation and other probabilistic causes.

**Assumption 1.** There exist real matrices \( G_1 \geq 0 \) and \( G_2 \geq 0 \) such that

\[
\text{trace}[\sigma^T(t, e(t), e(t - h(t))\sigma(t, e(t), e(t - h(t)))]
\]

\[
\leq e^T(t)G_1 e(t) + e^T(t - h(t))G_2 e(t - h(t)).
\]

Define the error signal \( e(t) = y(t) - x(t) \), then the error dynamic equation is given as follows:

\[
d[e(t) - Ce(t - h(t))] = [-Ae(t) + W_0 g(e(t)) + W_1 g(e(t - h(t))) + u(t)]dt
\]

\[
+ [\sigma(t, e(t), e(t - h(t))]d\omega(t),
\]

where \( g(e(t)) = f(x(t) + e(t)) - f(x(t)) \).

In this letter, the following feedback controller for synchronization between drive network (1) and response network (3) is proposed:

\[
u(t) = Ke(t),
\]

where \( K = \text{diag}\{k_1, k_2, \ldots, k_n\} \).
The following lemma and definition will be used in deriving the main result.

**Lemma 1.** For any constant matrix $M \in \mathbb{R}^{n \times n}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $x : [0, \gamma] \rightarrow \mathbb{R}^n$ such that the integrations concerned are well defined, then

$$
\left( \int_0^\gamma x(s)ds \right)^T M \left( \int_0^\gamma x(s)ds \right) \leq \gamma \int_0^\gamma x^T(s)Mx(s)ds.
$$

(7)

**Definition 1.** For the stochastic neural networks (5) and every $\phi \in L_2^2([-\bar{h}, 0], \mathbb{R}^n)$, the trivial solution is globally asymptotically stable in the mean square if

$$
\lim_{t \to \infty} \mathbb{E}[|x(t, \phi)|^2] = 0.
$$

(8)

In deriving our main results, Ito’s formula plays a key role in stability analysis of stochastic systems (see Refs. 27–29 for details).

3. Main Results

In this section, we propose a new criterion for synchronization of neural networks with stochastic perturbation described by Eqs. (1) and (3).

Now, we have the following main result.

**Theorem 1.** For given $h_d$, $\bar{h}$, and $L = \text{diag}\{l_1, l_2, \ldots, l_n\}$, the equilibrium point of (5) is globally asymptotically stable in the mean square if there exist positive scalars $\rho_1$ and $\rho_2$, positive diagonal matrices $P, Y, T_i$ ($i = 1, 2$), any diagonal matrix $S$, and positive definite matrices $Q, R_i$ ($i = 1, 2$) satisfying the following LMIs:

$$
\Psi = \begin{bmatrix}
\Psi_{11} & 0 & \Psi_{13} & \Psi_{14} & \Psi_{15} \\
* & \Psi_{22} & 0 & 0 & \Psi_{25} \\
* & * & \Psi_{33} & 0 & \Psi_{35} \\
* & * & * & \Psi_{44} & \Psi_{45} \\
* & * & * & * & \Psi_{55}
\end{bmatrix} < 0,
$$

(9)

$$
P \leq \rho_1 I, \quad Q \leq \rho_2 I,
$$

(10)

(11)

where

$$
\Psi_{11} = -SA - A^T S + 2Y + R_1 + \rho_1 G_1 + \bar{h}\rho_2 G_1 + L^T T_1 L,
$$

$$
\Psi_{13} = SW_0,
$$

$$
\Psi_{14} = SW_1,
$$

$$
\Psi_{15} = -A^T S + Y + P - S,
$$

$$
\Psi_{22} = \rho_1 G_2 - (1 - h_d)R_1 + \bar{h}\rho_2 G_2 + L^T T_2 L,
$$

$$
\Psi_{25} = -C^T P,
$$

$$
\Psi_{33} = R_2 - T_1,
$$

$$
\Psi_{35} = SW \Psi_{33} S,
$$

$$
\Psi_{44} = SW \Psi_{33} S
$$

$$
\Psi_{45} = SW \Psi_{33} S
$$

$$
\Psi_{55} = SW \Psi_{33} S
$$
Then, the control gain \( K \) of synchronization controller (6) is \( K = S^{-1}Y \).

**Proof.** System (1) can be represented as

\[
d[e(t) - Ce(t - h(t))] = q(t)dt + z(t)d\omega(t),
\]

where

\[
q(t) = -(A - K)e(t) + W_0g(e(t)) + W_1g(e(t - h(t))),
\]

\[
z(t) = \sigma(t, e(t), e(t - h(t))).
\]

Consider the following Lyapunov–Krasovskii functional candidate:

\[
V(e_t, t) = [e(t) - Ce(t - h(t))]^T P[e(t) - Ce(t - h(t))] + \int_{t-h(t)}^{t} e^T(s)R_1e(s)ds
\]

\[
+ \int_{t-h(t)}^{t} g^T(e(s))R_2g(e(s))ds + \int_{-\bar{h}}^{t} \int_{t+\theta}^{t} z^T(s)Qz(s)dsd\theta,
\]

where \( e_t = e(t + \theta), -\bar{h} \leq \theta \leq 0 \).

Then, by Ito’s formula, the stochastic differential \( dV(e_t, t) \) can be obtained as (see for example, Ref. 29),

\[
dV(e_t, t) = LV(e_t, t)dt + 2[e(t) - Ce(t - h(t))]^T P\sigma(t, e(t), e(t - h(t)))d\omega(t),
\]

where

\[
LV(e_t, t) = 2[e(t) - Ce(t - h(t))]^T Pq(t)
\]

\[
+ \text{trace}[\sigma^T(t, e(t), e(t - h(t)))P\sigma(t, e(t), e(t - h(t)))]
\]

\[
+ e(t)^T R_1e(t) - (1 - \bar{h}e(t))e^T(t - h(t))R_1e(t - h(t))
\]

\[
+ g(e(t))^T R_2g(e(t)) - (1 - \bar{h}e(t))g^T(e(t - h(t)))R_2g(e(t - h(t)))
\]

\[
+ \bar{h}z^T(t)Qz(t) - \int_{t-\bar{h}}^{t} z^T(s)Qz(s)ds.
\]

It follows from Eq. (4) that

\[
\text{trace}[\sigma^T(t, e(t), e(t - h(t)))P\sigma(t, e(t), e(t - h(t)))]
\]

\[
\leq \lambda_{\text{max}}(P)\text{trace}[\sigma^T(t, e(t), e(t - h(t)))\sigma(t, e(t), e(t - h(t)))]
\]

\[
\leq \rho_1[e^T(t)G_1e(t) + e^T(t - h(t))G_2e(t - h(t))],
\]

and

\[
\bar{h}z^T(t)Qz(t) \leq \bar{h}\rho_2[e^T(t)G_1e(t) + e^T(t - h(t))G_2e(t - h(t))].
\]
Using Lemma 1, we have
\[
dV(e_t, t) \leq \Sigma dt + 2[e(t) - Ce(t - h(t))]^T P \sigma(t, e(t), e(t - h(t)))d\omega(t),
\]
where
\[
\Sigma = 2[e(t) - Ce(t - h(t))]^T P q(t)
\]
\[
+ \rho_1[e^T(t)G_1e(t) + e^T(t - h(t))G_2e(t - h(t))]
\]
\[
+ e(t)^T R_1e(t) - (1 - h_d)e^T(t - h(t))R_1e(t - h(t))
\]
\[
+ g(e(t))^T R_2g(e(t)) - (1 - h_d)g^T(e(t - h(t)))R_2g(e(t - h(t)))
\]
\[
+ \bar{h}p_2[e^T(t)G_1e(t) + e^T(t - h(t))G_2e(t - h(t))] - \int_{t-h(t)}^t z^T(s)Qz(s)ds.
\]
From Eq. (13), the following equation with any diagonal matrix \(S\) holds:
\[
2[q^T(t)S + e^T(t)S][-q(t) - (A - K)e(t) + W_0g(e(t)) + W_1g(e(t - h(t)))] = 0.
\]
Here note that Eq. (2) means that
\[
g_j^2(e_j(t)) - l_j^2e_j^2(t) \leq 0 \quad (j = 1, \ldots, n),
\]
and
\[
g_j^2(e_j(t - h(t))) - l_j^2e_j^2(t - h(t)) \leq 0 \quad (j = 1, \ldots, n).
\]
From the two inequalities (20) and (21) above, for any diagonal positive matrices \(T_1 = \text{diag}\{t_{11}, \ldots, t_{1n}\}\) and \(T_2 = \text{diag}\{t_{21}, \ldots, t_{2n}\}\), the following inequalities hold:
\[
0 \leq - \sum_{j=1}^n t_{1j}[g_j^2(e_j(t)) - l_j^2e_j^2(t)] - \sum_{j=1}^n t_{2j}[g_j^2(e_j(t - h(t))) - l_j^2e_j^2(t - h(t))]
\]
\[
= e^T(t)L^TT_1Le(t) - g^T(e(t))T_1g(e(t))
\]
\[
+ e^T(t - h(t))L^TT_2Le(t - h(t)) - g^T(e(t - h(t)))T_2g(e(t - h(t))).
\]
For simplicity, let us define \(\zeta(t)\) as
\[
\zeta^T(t) = [e^T(t) \quad e^T(t - h(t)) \quad g^T(e(t)) \quad g^T(e(t - h(t))) \quad q^T(t)],
\]
and \(Y = SK\).
By utilizing relationships (18)–(22), we obtain that
\[
dV(e_t, t) \leq \Sigma_1 dt + 2[e(t) - Ce(t - h(t))]^T P \sigma(t, e(t), e(t - h(t)))d\omega(t),
\]
where
\[
\Sigma_1 = 2[e(t) - Ce(t - h(t))]^T P q(t)
\]
\[
+ \rho_1[e^T(t)G_1e(t) + e^T(t - h(t))G_2e(t - h(t))]
\]
\[
+ e(t)^T R_1e(t) - (1 - h_d)e^T(t - h(t))R_1e(t - h(t))
\]
\[
+ g(e(t))^T R_2g(e(t)) - (1 - h_d)g^T(e(t - h(t)))R_2g(e(t - h(t)))
\]
\[ + \dot{h}p_2[e^T(t)G_1e(t) + e^T(t - h(t))G_2e(t - h(t))] - \int_{t-h(t)}^{t} z^T(s)Qz(s)ds \\
+ 2q^T(t)[-Sq(t) - (SA - Y)e(t) + SW_0g(e(t)) + SW_1g(e(t - h(t)))] \\
+ 2e^T(t)[-Sq(t) - (SA - Y)e(t) + SW_0g(e(t)) + SW_1g(e(t - h(t)))] \\
+ e^T(t)L^T_1L(t) - g^T(e(t))T_1g(e(t)) \\
+ e^T(t - h(t))L^T_2L(t - h(t)) - g^T(e(t - h(t)))T_2g(e(t - h(t))) \\
= \left[ \zeta^T(t)\Psi\zeta(t) - \int_{t-h(t)}^{t} z^T(s)Qz(s)ds \right] \\
\leq \zeta^T(t)\Psi\zeta(t). \quad (25) \]

Note that if \( \Psi < 0 \), then there exists a positive scalar \( \gamma \) such that
\[ \Psi + \text{diag}\{\gamma I, 0, 0, 0, 0, 0, 0\} < 0. \quad (26) \]

By taking the mathematical expectation on both sides of Eq. (24) and considering (26), we have
\[ \frac{d}{dt}E[V(e(t),t)] \leq E[\zeta^T(t)\Psi\zeta(t)] \leq -\gamma E[|e(t)|^2], \quad (27) \]

which implies that the error dynamics (5) between master (1) and response network (3) is globally asymptotically stable in the mean square. This completes our proof.

**Remark 1.** The solutions of Theorem 1 can be obtained by solving the eigenvalue problem with respect to solution variables, which is a convex optimization problem. In this letter, we utilize Matlab’s LMI Control Toolbox which implements the interior-point algorithm. This algorithm is faster than classical convex optimization algorithms.

**Remark 2.** The work studied in Ref. 22 is a special case of our research with \( C = 0 \).

Now, we give two examples so as to demonstrate the effectiveness of our theoretical results.

**Example 1.** Consider the following stochastic neural networks studied in Ref. 22:
\[
A = \text{diag}\{1, 1\}, \quad W_0 = \begin{bmatrix} 2 & -0.1 \\ -5 & 3.2 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}, \quad C = 0, \quad G_1 = \text{diag}\{0.01, 0.05\}, \quad G_2 = \text{diag}\{0.5, 0.01\}, \\
L = I, \quad h = 1, \quad h_d = 0.
\]

Once again note that our control scheme is based on static feedback control to achieve synchronization of the system above, while an adaptive control scheme is
One can easily find that the LMIs given in Theorem 1 are feasible for any \( R \). Then, the control gain for synchronization is possible control gain is applied in Ref. 22. By applying Theorem 1 to the system and using Matlab’s LMI Toolbox,\(^{30}\) we have the following solutions:

\[
P = \text{diag}\{2.5515, 1.8784\}, \quad S = \text{diag}\{0.0822, 0.0143\}, \quad Y = \text{diag}\{-2.4989, -1.8537\},
\]

\[
T_1 = \text{diag}\{1.1111, 1.3563\}, \quad T_2 = \text{diag}\{0.4083, 0.6320\}, \quad Q = \text{diag}\{0.5877, 0.5877\},
\]

\[
R_1 = \begin{bmatrix}
2.8540 & 0.0026 \\
0.0026 & 1.3760 \\
\end{bmatrix}, \quad R_2 = \begin{bmatrix}
0.5251 & 0.0094 \\
0.0094 & 0.6366 \\
\end{bmatrix}, \quad \rho_1 = 2.9578, \quad \rho_2 = 1.1753.
\]

Then, the control gain for synchronization \( K = S^{-1}Y \) is

\[
K = \text{diag}\{-30.4064, -129.3023\}.
\]

By the way, in case of cellular networks without time delays, i.e., \( W_1 = G_2 = 0 \), one can easily see that the LMI solutions in Theorem 1 are also feasible. Then a possible control gain is

\[
K = \text{diag}\{-14.8156, -109.1130\}.
\]

**Example 2.** Consider the following three-dimensional stochastic neural networks with the following system matrices and parameters:

\[
A = \text{diag}\{3, 3, 2\}, \quad W_0 = \begin{bmatrix}
1 & 0.1 & 0 \\
0.1 & 0.3 & 0.2 \\
0.2 & 0.1 & 1 \\
\end{bmatrix}, \quad W_1 = \begin{bmatrix}
0.3 & 1 & -0.2 \\
0.1 & 0.2 & 0.1 \\
-0.2 & 0.1 & 0.5 \\
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0.2 & 0.05 & 0.1 \\
0.05 & 0.1 & 0.05 \\
0.05 & 0.1 & 0.2 \\
\end{bmatrix}, \quad G_1 = \text{diag}\{0.1, 0.2, 0.1\}, \quad G_2 = \text{diag}\{0.05, 0.02, 0.05\},
\]

\[
L = I, \quad h_d = 0.15.
\]

One can easily find that the LMIs given in Theorem 1 are feasible for any \( h \). For instance, when \( h = 10 \), the following solutions:

\[
P = 10^3 \times \text{diag}\{1.3138, 1.4253, 1.4097\}, \quad S = \text{diag}\{212.8511, 247.8902, 250.9645\},
\]

\[
Y = \text{diag}\{-660.9598, -631.2652, -883.0411\},
\]

\[
T_1 = \text{diag}\{716.1930, 743.5426, 718.6635\},
\]

\[
T_2 = \text{diag}\{160.1258, 341.9235, 179.6471\}, \quad Q = \text{diag}\{249.4045, 249.4045, 249.4045\},
\]

\[
R_1 = 10^3 \times \begin{bmatrix}
1.2002 & 0.0841 & 0.1479 \\
0.0841 & 1.0919 & 0.1250 \\
0.1479 & 0.1250 & 1.2990 \\
\end{bmatrix}, \quad R_2 = \begin{bmatrix}
222.2258 & -19.7719 & -75.7822 \\
-19.7719 & 444.7284 & -21.0595 \\
-75.7822 & -21.0595 & 254.5885 \\
\end{bmatrix},
\]

\[
\rho_1 = 10^3 \times 1.7552, \quad \rho_2 = 498.8091.
\]

Hence, the gain matrix \( K \) for controller (6) is as follows:

\[
K = \text{diag}\{-0.5031, -0.4429, -0.6264\}.
\]
References

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