Novel robust stability criterion for a class of neutral systems with mixed delays and nonlinear perturbations

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Abstract

In this paper, a class of neutral systems with mixed delays and nonlinear perturbations is considered. A novel robust stability criterion of the system is derived by using Lyapunov method. The criterion can be easily solved by efficient convex optimization algorithms. Some numerical examples are given to illustrate our result.

Keywords: Neutral systems; Nonlinear perturbations; Lyapunov method; Convex optimization

1. Introduction

Since time delay is frequently a source of instability and poor performance and commonly exists in various engineering, biological, and economical systems, the stability analysis of delay-differential systems has been widely investigated in the last decades [1,2]. Recently, the stability analysis of the delay systems of neutral type has been studied by some researchers [3–9,12]. Also, the problem is extended to neutral systems with perturbations by [10,11]. However, the stability criteria for operators used in their analysis is expressed in terms of matrix norm, which may lead to conservative result.

In this paper, we consider a class of neutral delay-differential systems with nonlinear perturbations:
\[
\frac{d}{dt}[x(t) - Cx(t - \tau_2)] = Ax(t) + Bx(t - \tau_1) + f_1(t,x(t)) + f_2(t,x(t - \tau_1)), \quad t \geq 0
\]
\[x(t_0 + \theta) = \phi(\theta), \quad \forall \theta \in [-\tau,0], \quad (1)\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(\tau_1, \tau_2 > 0\) are constant delays, \(\tau = \max(\tau_1, \tau_2)\), \(A, B\) and \(C\) are known constant real matrices of appropriate dimensions, \(\phi(\cdot) \in \mathcal{C}_0 : [-\tau,0] \rightarrow \mathbb{R}^n\) is the initial vector, and \(f_i(\cdot)\) is the nonlinear perturbations in the system. It is assumed that the nonlinear perturbations are bounded in magnitude as

\[
\|f_1(t,x(t))\| \leq z_1\|x(t)\|, \quad \|f_2(t,x(t - \tau_1))\| \leq z_2\|x(t - \tau_1)\|, \quad \forall t > 0,
\]

where \(z_1\) and \(z_2\) are positive scalars.

In this paper, using the Lyapunov functional technique combined with matrix inequality technique, we present a novel stability criterion for asymptotic stability of the system. The criterion will be derived in terms of matrix inequality. The matrix inequality can be easily solved by various efficient convex optimization algorithms [13].

2. Main results

Define an operator \(\mathcal{D}(\xi_t) : \mathcal{C}_0 \rightarrow \mathbb{R}^n\) as

\[
\mathcal{D}(\xi_t) = x(t) + B \int_{t-\tau_1}^{t} x(s) \, ds - Cx(t - \tau_2). \quad (2)
\]
Before proceeding further, we will state well known facts and lemmas.

**Fact 1.** For any vectors $a, b \in \mathbb{R}^n$ and scalar $\varepsilon > 0$,
\[ 2a^T b \leq \varepsilon a^T a + \varepsilon^{-1} b^T b. \]

**Fact 2.** The linear matrix inequality
\[ \begin{bmatrix} Z(x) & Y(x) \\ Y^T(x) & W(x) \end{bmatrix} > 0, \]
is equivalent to
\[ W(x) > 0, \quad Z(x) - Y(x) W^{-1}(x) Y^T(x) > 0, \]
where $Z(x) = Z^T(x)$, $W(x) = W^T(x)$ and $Y(x)$ depend affinely on $x$.

**Lemma 1** [12]. For given positive scalars $\tau_1$ and $\tau_2$, the operator $\mathcal{D}(x_i)$ is stable if there exist a positive-definite matrix $\Gamma$ and positive scalars $\beta_1$ and $\beta_2$ such that
\[ \beta_1 + \beta_2 < 1, \quad \begin{bmatrix} C^T \Gamma C - \beta_1 \Gamma & \tau_1 C^T \Gamma B \\ \tau_1^2 B^T \Gamma B - \beta_2 \Gamma \end{bmatrix} < 0. \quad (3) \]

**Lemma 2** [14]. For any constant matrix $\Phi \in \mathbb{R}^{n \times n}$, $\Phi = \Phi^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then
\[ \left( \int_0^\gamma \omega(s) \, ds \right)^T \Phi \left( \int_0^\gamma \omega(s) \, ds \right) \leq \gamma \int_0^\gamma \omega(s)^T \Phi \omega(s) \, ds. \]

Differentiating $\mathcal{D}(x_i)$ leads to
\[ \begin{align*}
\dot{\mathcal{D}}(x_i) &= \dot{x}(t) + Bx(t) - Bx(t - \tau_1) - Cx(t - \tau_2) \\
&= A_0 \left( \mathcal{D}(x_i) - B \int_{t-\tau_1}^t x(s) \, ds + Cx(t - \tau_2) \right) + f_1(t, x(t)) \\
&+ f_2(t, x(t - \tau_1)),
\end{align*} \quad (4) \]
where $A_0 = A + B$.

Now, we establish a criterion in terms of matrix inequality, for asymptotic stability of system (1) using the Lyapunov method.

**Theorem 1.** Given scalars $\tau_1 > 0$ and $\tau_2 > 0$, the system (1) is asymptotically stable if the operator $\mathcal{D}(x_i)$ is stable and there exist the positive-definite matrices $P$, $Z$, $W$, $Y$ and positive scalars $\varepsilon_1$ and $\varepsilon_2$ satisfying the following matrix inequality:
\[
\begin{bmatrix}
\alpha_1^TP + \alpha_1^B & \alpha_1^P & -\alpha_0^B & 0 & 0 & 0 & \alpha_1^C & \alpha_1^D & 0
\\
-\alpha_1^P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
\alpha_1^B & -\alpha_1^C & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
\alpha_1^C & \alpha_1^D & -\alpha_1^B & -\alpha_1^D & 0 & 0 & 0 & 0 & 0
\\
\alpha_1^D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
\alpha_1^B & -\alpha_1^C & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
\alpha_1^C & \alpha_1^D & -\alpha_1^B & -\alpha_1^D & 0 & 0 & 0 & 0 & 0
\\
\alpha_1^D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\\
\end{bmatrix} < 0.
\]

**Proof.** For \( P > 0, \ W > 0, \ Q > 0, \) and \( Y > 0, \) the functional given by

\[
V = V_1 + V_2 + V_3 + V_4,
\]  

where

\[
V_1 = \mathcal{D}(x_t)^T P \mathcal{D}(x_t),
\]  

\[
V_2 = \int_{t-\tau_2}^t x^T(s) Yx(s) \, ds,
\]  

\[
V_3 = \int_{t-\tau_1}^t (s - t + \tau_1) x^T(s) Qx(s) \, ds,
\]  

\[
V_4 = \int_{t-\tau_1}^t x^T(s) Wx(s) \, ds,
\]

is a legitimate Lyapunov functional candidate [1].

Taking the time derivative of \( V \) along the solution of (4) gives that

\[
\frac{dV_1}{dt} = 2 \mathcal{D}(x_t)^T \left[ A_0 \left( \mathcal{D}(x_t) - B \int_{t-\tau_1}^t x(s) \, ds + Cx(t - \tau_2) \right) \right.
\]

\[
+ f_1(t, x(t)) + f_2(t, x(t - \tau_1)) \bigg],
\]

\[
\leq 2 \mathcal{D}(x_t)^T \left[ A_0 \left( \mathcal{D}(x_t) - B \int_{t-\tau_1}^t x(s) \, ds + Cx(t - \tau_2) \right) \right.
\]

\[
+ \epsilon_1 \mathcal{D}(x_t)^T PP \mathcal{D}(x_t) + \epsilon_1^{-1} x_1^2 x_1^T(t) x(t) + \epsilon_2 \mathcal{D}(x_t)^T PP \mathcal{D}(x_t) + \epsilon_1^{-1} x_2^2 x_2^T(t - \tau_1) x(t - \tau_1),
\]

\[
\frac{dV_2}{dt} = x^T(t) \dot{y}(t) - x^T(t - \tau_2) \dot{y}(t - \tau_2),
\]
\[
\frac{dV_1}{dt} = \tau_1 x^T(t)Qx(t) - \int_{t-t_1}^{t} x^T(s)Qx(s)ds \leq \tau_1 x^T(t)Qx(t)
\]
\[
- \left( \int_{t-t_1}^{t} x(s)ds \right)^T \left( \tau_1^{-1}Q \right) \left( \int_{t-t_1}^{t} x(s)ds \right),
\]
(13)

\[
\frac{dV_2}{dt} = x^T(t)Wx(t) - x^T(t - \tau_1)Wx(t - \tau_1),
\]
(14)

where $\epsilon_1 > 0$, $\epsilon_2 > 0$, and Fact 1 and Lemma 2 are utilized in (11) and (12).

Let $M = \epsilon_1^{-1}x_1^2I + \tau_1Q + W + Y$ and note that

\[
x^T(t)Mx(t) = \left( \mathcal{D}(x_t) - B \int_{t-t_1}^{t} x(s)ds + Cx(t - \tau_2) \right)^T \times M \left( \mathcal{D}(x_t) - B \int_{t-t_1}^{t} x(s)ds + Cx(t - \tau_2) \right),
\]
(15)

Using (11)–(15), we can obtain a new bound of the time derivative of $V$ as

\[
\frac{dV}{dt} = \sum_{i=1}^{4} \frac{dV_i}{dt} \leq \chi^T(t)\Omega_0\chi(t),
\]
(16)

where

\[
\chi(t) = \begin{bmatrix}
\mathcal{D}(x_t) \\
\int_{t-t_1}^{t} x(s)ds \\
x(t - \tau_1) \\
x(t - \tau_2)
\end{bmatrix},
\]
(17)

and

\[
\Omega_0 = \begin{bmatrix}
A_0^T P + P A_0 & -P A_0 B - M B & 0 & P A_0 C \\
+M + \epsilon_1PP + \epsilon_2PP & -\tau_1^{-1}Q + B^T M B & 0 & -B^T M C \\
\star & \star & -W + \epsilon_1^2x_1^2I & 0 \\
\star & \star & \star & -Y + C^T M C
\end{bmatrix}
\]
(18)
Therefore, if $\Omega_0 < 0$, there exists the positive scalar $\delta$ such that
\[
\frac{dV}{dt} \leq -\delta \|X_i\|^2.
\] (19)

By Fact 2, the inequality $\Omega_0 < 0$ in (18) is equivalent to
\[
\Omega_1 = \begin{bmatrix}
\Psi & -PA_0B & 0 & PA_0C & \tau_1I & I & \epsilon_1^2I & I \\
-\tau_1^{-1}Q & 0 & 0 & -\tau_1B^T & -B^T & -\epsilon_1^{-1}z_1B^T & -B^T \\
0 & -W + \epsilon_1^{-1}z_1^2I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\tau_1Q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -W^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & -e_1^{-1}I & 0 & -Y^{-1} & 0 \\
0 & 0 & 0 & 0 & -Y^{-1} & 0 & 0
\end{bmatrix} < 0,
\] (20)

where $\Psi = A_0^T P + PA_0 + \epsilon_1 PP + \epsilon_2 PP$.

Letting $Z = \tau_1^{-1}Q$, and pre- and post-multiplying the matrix $\Omega_1$ by diag\{I, I, I, Z, W, $e_1I$, $Y$\}, give that $\Omega_1 < 0$ is equivalent to the following inequality:
\[
\begin{bmatrix}
\Psi & -PA_0B & 0 & PA_0C & \tau_1Z & W & z_1I & Y \\
-\tau_1I & 0 & 0 & -\tau_1B^T Z & -B^T W & -\epsilon_1^{-1}z_1B^T & -B^T Y \\
0 & -W + \epsilon_1^{-1}z_1^2I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\tau_1Q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -W^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & -e_1^{-1}I & 0 & -Y^{-1} & 0 \\
0 & 0 & 0 & 0 & -Y^{-1} & 0 & 0
\end{bmatrix} < 0.
\] (21)

Again, by Fact 2, the inequality (21) is equivalent to the matrix inequality (5). This implies that both the system (1) and (4) with stable operator $\mathcal{D}(x_i)$ are asymptotically stable by Theorem 9.8.1 in [1].

This completes our proof. \qed

Remark 1. The inequality problem of Theorem 1 is to determine whether the problem is feasible or not. It is called the feasibility problem. The solutions of the problem can be found by solving generalized eigenvalue problem in $P$, $Z$, $W$, $Y$, $e_1$, and $e_2$, which is a quasiconvex optimization problem. Note that a locally optimal point of a quasiconvex optimization problem with strictly quasiconvex objective is globally optimal. For details, see Boyd et al. [13]. Various efficient convex optimization algorithms can be used to check whether the matrix inequality (5) is feasible. In this paper, in order to solve the matrix inequality, we utilize Matlab’s LMI Control Toolbox [15], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [13].
Remark 2. To guarantee less conservatism of stability criteria for operator $\mathcal{D}(x_t)$, Lemma 2 is utilized in the paper. However, the criteria for stability of certain operator are expressed in terms of matrix norm in the works of [9–11], which is generally more conservative. Also, the result of [11] is only applicable to the system with $\tau_1 = \tau_2$.

Remark 3. Note that the criterion of Theorem 1 is delay-dependent and delay-independent with respect to $\tau_1$ and $\tau_2$, respectively. The maximum allowable bound on the delay $\tau_1$ for guaranteeing stability of system (1) can be obtained by solving iteratively the matrix inequality (5) with respect to $\tau_1$.

3. Numerical examples

Example 1. Consider the neutral system without nonlinear perturbations:

$$\frac{d}{dt}[x(t) - Cx(t - \tau_2)] = Ax(t) + Bx(t - \tau_1),$$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.2 \end{bmatrix}.$$

Using the approaches in [3,5,9–11], no conclusion can be made since their stability condition for certain operators are not satisfied. However, by applying Theorem 1, one can see that the system is asymptotically stable for $\tau_1 < 0.887$.

For reference, the solutions of the matrix inequality (5) for $\tau_1 = 0.887$ are as follows:

$$P = \begin{bmatrix} 19.9630 & 33.9813 \\ 33.9813 & 108.8896 \end{bmatrix}, \quad Z = \begin{bmatrix} 32.7109 & 7.1834 \\ 7.1834 & 23.7107 \end{bmatrix},$$

$$Y = \begin{bmatrix} 2.5937 & 8.3000 \\ 8.3000 & 72.7121 \end{bmatrix}, \quad W = \begin{bmatrix} 0.5794 & -0.9797 \\ -0.9797 & 1.7885 \end{bmatrix},$$

$$\epsilon_1 = 9.4073 \times 10^3, \quad \epsilon_2 = 1.0906 \times 10^4,$$

$$\Gamma = \begin{bmatrix} 530.3482 & 6.7251 \\ 6.7251 & 555.0094 \end{bmatrix},\quad \beta_1 = 0.3333, \quad \beta_2 = 0.3333.$$

Example 2. Consider the neutral system:

$$\frac{d}{dt}[x(t) - Cx(t - \tau)] = Ax(t) + Bx(t - \tau) + f_1(t,x(t)) + f_2(t,x(t - \tau)),$$
\[ \begin{bmatrix} -2 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0.4 \\ 0.4 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0.2 & 1 \\ 0 & 0.2 \end{bmatrix}, \]

\[ f_1(t, x(t)) = (\delta_1 \cos t|x_1(t)|, \delta_2 \sin t|x_1(t)|)^T, \]

\[ f_2(t, x(t-\tau)) = (\gamma \cos t|x_1(t-\tau)|, \gamma_2 \sin t|x_1(t-\tau)|)^T, \]

where \(|\delta_i| \leq \alpha_1 = 0.2, |\gamma_i| \leq \alpha_2 = 0.1 (i = 1, 2)\).

Clearly, we have

\[ \|f_1(t, x(t))\| \leq \alpha_1\|x(t)\|, \quad \|f_2(t, x(t-\tau))\| \leq \alpha_2\|x(t-\tau)\|. \]

Applying Theorem 1 to the system gives that the stability bound on \(\tau\) is \(\tau < 0.583\).

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References

