On robust $\mathcal{H}_\infty$ filter design for uncertain neural systems: LMI optimization approach

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Abstract

This article deals with the robust $\mathcal{H}_\infty$ filtering problem for neutral delay differential systems with parametric uncertainties. A linear matrix inequality (LMI) approach is proposed to design the robust $\mathcal{H}_\infty$ filter such that the filtering system remains asymptotically stable and the bound of $\mathcal{H}_\infty$ norm is minimized. The Lyapunov stability theory is used for analysis of the system. A numerical example is provided to illustrate the validity of proposed design approach.

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1. Introduction

State estimation has been one of the fundamental issues in the control area. Recently, there has been a lot of interest on the problem of robust $\mathcal{H}_\infty$ filtering for dynamic systems with parametric uncertainties. In the $\mathcal{H}_\infty$ filtering, the
exogenous noise signal is assumed to be energy bounded rather than Gaussian, and the problem is to design a filter such that the $H_1$ norm of the system, which reflects the worst-case gain of the system, is minimized [11].

On the other hand, the stability analysis and stabilization problem for delay differential systems has been received considerable attention during the last few decades [1,2]. It is well-known that the delay is often main cause of instability and poor performance of systems. More recent years, the stability analysis for neutral delay-differential equations/systems of various types has been extensively studied [3–9]. Various methods has been introduced to derive less conservative and concise stability criteria for neutral systems [3–8]. A control scheme to guarantee the adequate level of performance has been presented by Park [9], and the design problem on observer-based controller of a class of neutral system is investigated in Park [10]. However, so far, the robust $H_\infty$ filtering problem for neutral differential systems has not been fully investigated and remains to be important and challenging.

In this paper, the problem of robust $H_\infty$ filtering for neutral delay-differential systems subjected to parameter uncertainties is investigated. The uncertainties are assumed to be bounded. Using the Lyapunov functional technique combined with LMI technique, we develop a robust $H_\infty$ filter for this system, which makes the closed-loop system asymptotically stable and the bound of $H_\infty$ norm be minimized. A stability criterion for the existence of the filter is derived in terms of LMIs, and theirs solutions provide a parameterized representation of the filter. The LMIs can be easily solved by various efficient convex optimization algorithms [13].
2. Problem formulation

Consider a class of neutral delay-differential system of the form:

\[
\begin{align*}
\frac{d}{dt} [x(t) - A_2 x(t - \tau)] &= (A_0 + \Delta A_0(t))x(t) + (A_1 + \Delta A_1(t))x(t - h) + Bw(t), \\
y(t) &= (C + \Delta C(t))x(t) + Dw(t), \\
x(t_0 + \theta) &= \phi(\theta), \quad \forall \theta \in [-H, 0],
\end{align*}
\]

(1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( A_0, A_1, A_2, B, C, \) and \( D \) are known constant real matrices of appropriate dimensions, \( w(t) \in \mathbb{R}^m \) is the noise signal vector (including process and measurement noises) in \( L_2 \), \( y(t) \in \mathbb{R}^p \) is the measured output, \( h \) and \( \tau \) are the positive constant time delays, \( \Delta A_0(t), \Delta A_1(t) \) and \( \Delta C(t) \) are time-varying parameter uncertainties, \( H = \max\{h, \tau\}, \) and \( \phi(\cdot) \in \mathcal{C}_0 : [-H, 0] \rightarrow \mathbb{R}^n \) is the initial vector. The admissible uncertainties are assumed to be of the form

\[
\begin{align*}
\Delta A_0(t) &= D_0 F_0(t) E_0, \\
\Delta A_1(t) &= D_1 F_1(t) E_1, \\
\Delta C(t) &= D_2 F_2(t) E_2,
\end{align*}
\]

(2)

where \( D_0, D_1, D_2, E_0, E_1 \) and \( E_2 \) are real constant matrices with appropriate dimensions, and \( F_0(t), F_1(t) \) and \( F_2(t) \) are unknown perturbation matrices with Lebesgue measurable elements and satisfies

\[
\|F_0(t)\| \leq 1, \quad \|F_1(t)\| \leq 1, \quad \|F_2(t)\| \leq 1.
\]

The robust filter design problem tackled in this paper consists of obtaining an estimate \( \tilde{z}(t) \) of the signal \( x(t) \) that provides small estimation error \( e(t) = x(t) - \tilde{z}(t) \) for all \( w \in L_2 \) and uncertainties. Attention is focused on the design of a time-invariant stable filter of order \( n \) with state-space realization of the form

\[
\begin{align*}
\frac{d}{dt} [\tilde{z}(t) - A_2 \tilde{z}(t - \tau)] &= G \tilde{z}(t) + A_1 \tilde{z}(t - h) + Ky(t), \\
\tilde{z}(t_0 + \theta) &= 0, \quad \forall \theta \in [-H, 0],
\end{align*}
\]

(3)

where \( G \) and \( K \) are the filter parameter to be determined later.

Then, it follows that the error dynamics is given by

\[
\begin{align*}
\dot{e}(t) &= \dot{x}(t) - \dot{\tilde{z}}(t) \\
&= Ge(t) + (A_0 + \Delta A - K(C + \Delta C) - G)x(t) + A_1 e(t - h) \\
&\quad + \Delta A_1 x(t - h) + A_2 \dot{e}(t - \tau) + (B - KD)w(t)
\end{align*}
\]

(4)
and the output of error is defined as
\[ y_e(t) = Le(t), \]  
where L is a known constant matrix.

Thus, by (1) and (4), it is easy to obtain the augmented system
\[
\frac{d}{dt}[z(t) - \tilde{A}_2z(t - \tau)] = (\tilde{A}_0 + \Delta \tilde{A}_0)z(t) + (\tilde{A}_1 + \Delta \tilde{A}_1)z(t - h) + \bar{B}w(t),
\]
\[ y_e(t) = \bar{L}z(t), \]

where

\[
\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ A_0 - KC - G & G \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix},
\]
\[
\tilde{B} = \begin{bmatrix} B \\ B - KD \end{bmatrix}, \quad \bar{L} = [0 \ L], \quad z(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix},
\]
\[
\Delta \tilde{A}_0(t) = \begin{bmatrix} \Delta A_0 & 0 \\ \Delta A_0 - K\Delta C & 0 \end{bmatrix} = \begin{bmatrix} D_0 \\ 0 \end{bmatrix} F_0(t)[E_0 \ 0] + \begin{bmatrix} 0 \\ D_0 \end{bmatrix} F_0(t)[E_0 \ 0]
\]
\[
+ \begin{bmatrix} 0 \\ -KD_2 \end{bmatrix} F_2(t)[E_2 \ 0]
\]
\[
= \tilde{D}_0 F_0(t)E_0 + \tilde{D}_0 F_0(t)\tilde{E}_0 + \tilde{D}_2 F_2(t)\tilde{E}_2,
\]
\[
\Delta \tilde{A}_1(t) = \begin{bmatrix} 0 & 0 \\ \Delta A_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ D_1 \end{bmatrix} F_1(t)[E_1 \ 0] \equiv \tilde{D}_1 F_1(t)\tilde{E}_1.
\]

Let \( T_{zw} \) denote a stable operator from signal \( w(t) \) to signal \( z(t) \). Then \( \mathcal{H}_\infty \) norm of \( T_{zw} \) is defined as
\[
\|T_{zw}\|_\infty = \sup_{w(t) \in \mathcal{F}_2} \frac{\|z(t)\|_2}{\|w(t)\|_2}.
\]

Here, we introduce \( \mathcal{H}_\infty \) performance measure for system (6) as follows:
\[
J = \int_0^\infty [y_e^T(t)y_e(t) - \gamma^2 w^T(t)w(t)] dt.
\]

Therefore, the problem tackled in this paper is to determine the filter (3) within the upper bound, i.e,
\[
\|T_{yzw}\|_\infty = \sup_{w(t)} \frac{\|y_e(t)\|_2}{\|w(t)\|_2} < \gamma.
\]

In this situation, the filtering error system is said to have a guaranteed \( \gamma \) level of noise attenuation. In next section, the criterion for minimization of \( \gamma \) level will be derived using LMI convex optimization approach.
3. Robust stability analysis and $H\infty$ filter design

Before proceeding further, we will give two lemmas and a fact.

**Lemma 1** [14]. For any constant symmetric positive-definite matrix $\Theta \in \mathbb{R}^{m \times m}$, a scalar $\sigma > 0$, and the vector function $\omega : [0, \sigma] \rightarrow \mathbb{R}^n$ such that the integrations in the following are well defined, then

$$\sigma \int_0^\sigma \omega^T(s)\Theta\omega(s)\,ds \geq \left(\int_0^\sigma \omega(s)\,ds\right)^T \Theta \left(\int_0^\sigma \omega(s)\,ds\right).$$

**Lemma 2** [12]. For given positive scalars $h$ and $\tau$ and any $E_1, E_2 \in \mathbb{R}^{n \times n}$, the operator $\mathcal{D}(x_t) : \mathbb{C}_0 \rightarrow \mathbb{R}^n$ defined by

$$\mathcal{D}(x_t) = x(t) + E_1 \int_{t-h}^t x(s)\,ds - E_2 x(t - \tau)$$

is stable if there exist a positive definite matrix $\Gamma$ and positive scalars $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1 + \alpha_2 < 1,$$

$$\begin{bmatrix} E_1^T \Gamma E_2 - \alpha_1 \Gamma & hE_1^T \Gamma E_1 \\ \star & h^2 E_1^T \Gamma E_1 - \alpha_2 \Gamma \end{bmatrix} < 0.$$  

**Fact 1** (Schur complement). Given constant symmetric matrices $\Sigma_1$, $\Sigma_2$, $\Sigma_3$ where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

Here, we define an operator $\mathcal{D}(z_t) : \mathbb{C}_0 \rightarrow \mathbb{R}^n$

$$\mathcal{D}(z_t) = z(t) + \bar{A}_1 \int_{t-h}^t z(s)\,ds - \bar{A}_2 z(t - \tau),$$

$$z_t = z(t + \theta), \quad \forall \quad \theta \in [-h, 0].$$

**Remark 1.** By Lemma 2, the operator $\mathcal{D}(z_t)$ is stable if there exist a positive definite matrix $\Gamma$ and positive scalars $\alpha_1$ and $\alpha_2$ such that

$$\begin{bmatrix} \bar{A}_2^T \Gamma \bar{A}_2 - \alpha_1 \Gamma & h\bar{A}_1^T \Gamma \bar{A}_1 \\ \star & h^2 \bar{A}_1^T \Gamma \bar{A}_1 - \alpha_2 \Gamma \end{bmatrix} < 0, \quad \text{and} \quad \alpha_1 + \alpha_2 < 1.$$
Differentiating $\mathcal{D}(z_t)$ and combining Eq. (6) leads to
\[
\dot{\mathcal{D}}(z_t) = (\tilde{A} + \Delta A_0(t))z(t) + \Delta A_1(t)z(t - h) + Bw(t)
\]
\[
= (\tilde{A} + \Delta A_0(t))\mathcal{D}(z_t) + \Delta A_1(t)z(t - h) + Bw(t)
\]
\[
- (\tilde{A} + \Delta A_0(t))\tilde{A}_1 \int_{t-h}^t z(s) \, ds - (\tilde{A} + \Delta A_0(t))\tilde{A}_2 z(t + \tau),
\]
(13)

where $\tilde{A} = A_0 + \tilde{A}_1$.

Now, we have the first main result of the paper.

**Theorem 1.** Consider the system (6) with $w(t) = 0$. For given scalars $h > 0$ and $\tau > 0$, suppose that there exist $\Gamma > 0$, $z_1 > 0$, and $x_2 > 0$ satisfying (12). If there exist positive scalars $\epsilon_i$ for $i = 0, 1, \ldots, 9$, $P > 0$, $Q_1 > 0$, $Q_2 > 0$, and $Q_3 > 0$ satisfying the following LMI:

\[
\begin{bmatrix}
\Sigma_1 & \Sigma_2 & 0 & PA\tilde{A}_2 + \tilde{Q}\tilde{A}_2 & -hPA\tilde{A}_1 - h\tilde{Q}\tilde{A}_1 \\
\Sigma_3 & 0 & 0 & 0 & 0 \\
\Sigma_4 & 0 & 0 & 0 & 0 \\
\Sigma_5 & \Sigma_6 & -h\tilde{A}_1^T\tilde{Q} & \Sigma_7 & \Sigma_8 \\
\end{bmatrix} < 0,
\]

(14)

where
\[
\Sigma_1 = \tilde{A}^T P + PA + \tilde{Q} + \epsilon_0 E_0^T E_0 + \epsilon_1 \tilde{E}_0^T E_0 + \epsilon_2 \tilde{E}_2^T E_2,
\]
\[
\Sigma_2 = \begin{bmatrix} P\tilde{D}_{01} & P\tilde{D}_{02} & P\tilde{D}_1 & P\tilde{D}_{01} & P\tilde{D}_{02} & hP\tilde{D}_{01} & hP\tilde{D}_{02} & hP\tilde{D}_2 \end{bmatrix},
\]
\[
\Sigma_3 = \text{diag}\{-\epsilon_0 I, -\epsilon_1 I, -\epsilon_2 I, -\epsilon_3 I, -\epsilon_4 I, -\epsilon_5 I, -\epsilon_6 I, -\epsilon_7 I, -\epsilon_8 I, -\epsilon_9 I\},
\]
\[
\Sigma_4 = -Q_2 + \epsilon_1 \tilde{E}_1^T E_1 + \tilde{A}_2^T \tilde{Q} \tilde{A}_2,
\]
\[
\Sigma_5 = -Q_2 + \epsilon_4 \tilde{E}_2^T E_0 \tilde{E}_0 \tilde{A}_2 + \epsilon_6 \tilde{E}_2^T E_2 \tilde{A}_2 + \epsilon_8 \tilde{E}_2^T E_2 \tilde{A} \tilde{A}_2,
\]
\[
\Sigma_6 = -hQ_1 + \epsilon_7 \tilde{E}_1^T E_0 \tilde{E}_0 \tilde{A}_1 + \epsilon_8 \tilde{E}_1^T E_2 \tilde{E}_0 \tilde{A}_1 + \epsilon_9 \tilde{E}_1^T E_2 \tilde{E}_2 \tilde{A}_1 + h^2 \tilde{Q},
\]
\[
\tilde{Q} = hQ_1 + Q_2 + Q_3,
\]

then, the filtering error system (6) is asymptotically stable.

**Proof.** Consider the following legitimate Lyapunov functional candidate [1]:
\[
V(t, z(t)) = V_1 + V_2 + V_3 + V_4,
\]
(15)

where
\[
V_1 = \mathcal{D}^T(z_t)P\mathcal{D}(z_t), \quad V_2 = \int_{t-h}^t (s - t + h)z^T(s)Q_1z(s) \, ds,
\]
\[
V_3 = \int_{t-h}^t z^T(s)Q_2z(s) \, ds, \quad V_4 = \int_{t-\tau}^t z^T(s)Q_3z(s) \, ds,
\]

and $P > 0$, $Q_1 > 0$, $Q_2 > 0$, and $Q_3 > 0$. 

Taking the time derivative of $V$ along the solution of (13) gives that
\[
\frac{dV_1}{dt} = 2 \left[ (\vec{A} + \Delta \vec{A}_0(t))\Omega(z_i) + \Delta \vec{A}_1(t)z(t-h) \\
- (\vec{A} + \Delta \vec{A}_0(t)) \int_{t-h}^{t} z(s) \, ds - (\vec{A} + \Delta \vec{A}_0(t)) \vec{A}_2 z(t-\tau) \right]^T P \Omega(z_i),
\]
(16)

\[
\frac{dV_2}{dt} = h z^T(t)Q_1 z(t) - \int_{t-h}^t z^T(s)Q_1 z(s) \, ds \leq h z^T(t)Q_1 z(t) \\
- \left( \frac{1}{h} \int_{t-h}^{t} z(s) \, ds \right)^T \left( \frac{1}{h} \int_{t-h}^{t} z(s) \, ds \right),
\]
(17)

\[
\frac{dV_3}{dt} = z^T(t)Q_2 z(t) - z^T(t-h)Q_2 z(t-h),
\]
(18)

\[
\frac{dV_4}{dt} = z^T(t)Q_3 z(t) - z^T(t-\tau)Q_3 z(t-\tau),
\]
(19)

where Lemma 1 is utilized in (17).

Here note that
\[
z^T(t)Q_i z(t) = \Omega^T(z_i)Q_i \Omega(z_i) - 2 \Omega^T(z_i)Q_i \vec{A}_1 \int_{t-h}^{t} z(s) \, ds + 2 \Omega^T(z_i)Q_i \vec{A}_2 z(t-\tau) \\
+ \left( \vec{A}_1 \int_{t-h}^{t} z(s) \, ds \right)^T Q_i \left( \vec{A}_1 \int_{t-h}^{t} z(s) \, ds \right) \\
- 2 \left( \vec{A}_1 \int_{t-h}^{t} z(s) \, ds \right)^T Q_i \vec{A}_2 z(t-\tau) + z^T(t-\tau) \vec{A}_2^T Q_i \vec{A}_2 z(t-\tau).
\]

Then, we have
\[
\frac{dV}{dt} = \sum_{i=1}^{4} \frac{dV_i}{dt} \leq \chi^T(t)\Omega \chi(t),
\]
(20)

where $\chi^T(t) = [\Omega^T(z_i)z^T(t-h)z^T(t-\tau)(h^{-1} \int_{t-h}^{t} z(s) \, ds)^T]$ and
\[
\Omega = \begin{bmatrix}
(\vec{A} + \Delta \vec{A}_0(t))^T P + P(\vec{A} + \Delta \vec{A}_0(t)) \\
\vec{Q} + \vec{A}_1 \vec{Q} \vec{A}_2 \\
\star \quad -Q_2 + \vec{A}_2^T \vec{Q} \vec{A}_2 \\
\star \quad \star \quad -Q_3 + h \vec{A}_2^T \vec{Q} \\
\star \quad \star \quad \star \quad -hQ_1 + h^2 \vec{Q}
\end{bmatrix},
\]
(21)

where $\Omega_{1,2} = P(\vec{A} + \Delta \vec{A}_0(t)) \vec{A}_2 + \vec{Q} \vec{A}_2$ and $\Omega_{1,3} = -hP(\vec{A} + \Delta \vec{A}_0(t)) \vec{A}_1 - h \vec{Q} \vec{A}_1$. 
Therefore, if $\Omega < 0$, there exists the positive scalar $\gamma$ such that
\[
\frac{dV}{dt} \leq -\gamma \|z(t)\|^2.
\] (22)

Using the well-known inequality
\[
\pm 2a^T b \leq \frac{1}{e} a^T a + e b^T b
\] (23)
for any $a, b \in \mathbb{R}^n$ and scalar $e > 0$, one can eliminate the uncertain factors $F_i(t)$ which are included in $\Delta A_0(t)$ and $\Delta A_1(t)$. Thus, we get the following relation:
\[
\Omega \leq \begin{bmatrix}
\Omega_1 & 0 & P\bar{A}A_2 + \bar{Q}A_2 & -hP\bar{A}A_1 - h\bar{Q}A_1 \\
\star & \Omega_2 & 0 & 0 \\
\star & \star & \Omega_3 & -h\bar{A}_1^T \bar{Q} \\
\star & \star & \star & \Omega_4
\end{bmatrix} \equiv \bar{\Omega},
\] (24)
where
\[
\begin{align*}
\Omega_1 &= \bar{A}_1^T P + P\bar{A} + \bar{Q} + e_0^{-1}PD_0D_0^T P + e_0\bar{E}_0^T \bar{E}_0 \\
&\quad + e_1^{-1}PD_02D_0^T P + e_1\bar{E}_0^T \bar{E}_0 + e_2^{-1}PD_2D_2^T P + e_2\bar{E}_2^T \bar{E}_2 + e_3^{-1}PD_1D_1^T P \\
&\quad + e_4^{-1}PD_02D_2^T P + e_4\bar{E}_2^T \bar{E}_2 + e_5^{-1}PD_2D_1^T P + e_5\bar{E}_2^T \bar{E}_0 + e_6^{-1}PD_1D_0^T P \\
&\quad + e_6\bar{E}_0^T \bar{E}_0 + e_7^{-1}hPD_0D_0^T P + e_7 h\bar{E}_0^T \bar{E}_0 + h^2\bar{Q},
\end{align*}
\]
\[
\Omega_2 = -\bar{Q}_2 + e_3\bar{E}_1^T \bar{E}_1 + \bar{A}_2^T \bar{Q}A_2,
\]
\[
\Omega_3 = -\bar{Q}_3 + e_4\bar{A}_1^T \bar{E}_0^T \bar{E}_0 A_2 + e_5\bar{A}_2^T \bar{E}_0^T \bar{E}_0 A_2 + e_6\bar{A}_2^T \bar{E}_2^T \bar{E}_2 A_2,
\]
\[
\Omega_4 = -h\bar{Q}_4 + e_7\bar{A}_1^T \bar{E}_0^T \bar{E}_0 A_1 + e_8\bar{A}_1^T \bar{E}_0^T \bar{E}_0 A_1 + e_9\bar{A}_1^T \bar{E}_2^T \bar{E}_2 A_1 + h^2\bar{Q}.
\]

By Fact 1 (Schur complement), the negative-definiteness of $\bar{\Omega}$ is equivalent to the LMI (14). Thus, the feasibility of the LMI (14) guarantees the negativity (22) of $\frac{dV}{dt}$. Therefore, by Theorem 9.8.1 (pp. 292–293) of Hale and Lunel [1] with the stable operator $\mathcal{D}(z)$ and (22), we conclude that system (1) and (6) are both asymptotically stable. □

Next, the following theorem provides a criterion for the existence of a linear stable filter assuring a robust $\mathcal{H}_\infty$ performance for neutral systems (1).

**Theorem 2.** Let $A = A_0 + A_1$ and $\rho = \gamma^2$. For $w(t) \in \mathcal{L}_2$ and given delays $h$ and $\tau$, consider the system (6). If the following optimization problem:
minimize \( \rho \)

subject to

\[
\begin{bmatrix}
    P_1 & P_2 & P_4 & 0 & 0 & 0 \\
    \star & \star & \star & \star & \star & \star \\
    \star & \star & \star & \star & \star & \star \\
    \star & \star & \star & \star & \star & \star \\
    \star & \star & \star & \star & \star & \star \\
    \star & \star & \star & \star & \star & \star \\
\end{bmatrix}
\begin{bmatrix}
    P_{11} \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    (P_2A_0 - M_2C - M_1)A_2 \\
    -hA_2^T(R_2 + R_5 + R_8) \\
    -h(R_2 + R_5 + R_8) \\
    -h(R_2 + R_5 + R_8) \\
    -h(R_2 + R_5 + R_8) \\
    -hR_2 + h^2(R_2 + R_5 + R_8) \\
    -hR_3 + h^2(R_3 + R_6 + R_9) \\
\end{bmatrix}
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    P_{12} \\
    0 \\
    0 \\
    0 \\
    0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    \star \\
    \star \\
    \star \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    R_1 & R_2 \\
    \star & R_3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    R_4 & R_5 \\
    \star & R_6 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    R_7 & R_8 \\
    \star & R_9 \\
\end{bmatrix}
\]

has the positive definite solutions, \( P_1, P_2, R_1, R_3, R_4, R_6, R_7, R_9, \) positive scalars \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_9, \rho, \) and matrices \( M_1, M_2, R_2, R_5, R_8, \) then the filtering error system has a guaranteed \( \gamma \) level of noise attenuation by robust \( \mathcal{H}_\infty \) filter (3).
Here, the following notations are defined:

\[
\begin{align*}
\Pi_1 &= A^T P_1 + P_1 A + R_1 + R_4 + R_7 + \varepsilon_6 E_1^1 E_0 + \varepsilon_1 E_1^0 E_0 + \varepsilon_2 E_1^T E_2, \\
\Pi_2 &= [P_1 D_0 \quad P_1 D_0 \quad hP_1 D_0], \\
\Pi_3 &= \text{diag} \{-\varepsilon_6 I, -\varepsilon_4 I, -h_5 I\}, \\
\Pi_4 &= A_0^T P_2 - C^T M_2^T - M^T + R_2 + R_3 + R_8, \\
\Pi_5 &= M_1 + M^T_1 + P_2 A_1 + A_1^T P_2 + R_3 + R_6 + R_9 + L^T L, \\
\Pi_6 &= [P_2 D_0 \quad M_2 D_2 \quad P_2 D_0 \quad M_2 D_2 \quad hP_2 D_0 \quad hM_2 D_2], \\
\Pi_7 &= \text{diag} \{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -h_6 I, -h_9 I\}, \\
\Pi_8 &= -R_4 + \varepsilon_3 E_1^1 E_1 + A_2^T (R_1 + R_4 + R_5) A_2, \\
\Pi_9 &= -R_5 + A_2^T (R_2 + R_3 + R_8) A_2, \\
\Pi_{10} &= -R_6 + A_2^T (R_3 + R_6 + R_9) A_2, \\
\Pi_{11} &= -R_7 + \varepsilon_4 A_2^T E_0^1 E_0^1 A_2 + \varepsilon_5 A_2^T E_0^1 E_0^1 A_2 + \varepsilon_6 A_2^T E_2^T E_2 A_2, \\
\Pi_{12} &= -hR_1 + h^2 (R_1 + R_4 + R_7) + \varepsilon_7 h A_2^T E_0^1 E_0^1 A_1 + \varepsilon_8 h A_2^T E_0^1 E_0^1 A_1 + \varepsilon_9 h A_2^T E_2^T E_2 A_1.
\end{align*}
\]

\(27\)

**Proof.** To establish the \(\mathcal{H}_\infty\) performance for the filtering error system, first notice that since the system (6) is asymptotically stable, then \(z(t)\) tends to zero as \(t \to \infty\). Next, assuming zero initial conditions for the filtering error system, the performance index is

\[
J = \int_0^\infty [y_1^T(t)y_1(t) - \gamma^2 w^T(t)w(t)] \, dt
\]

\[
= \int_0^\infty [y_1^T(t)y_1(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t,z(t))] \, dt
\]

\[
+ V(t,z(t))|_{t=0} - V(t,z(t))|_{t\to\infty},
\]

where \(V(t,z(t))\) is as in (15).

Considering that \(V(t,z(t))|_{t=0} = 0\) and \(V(t,z(t))|_{t\to\infty} \to 0\), one can easily infer from (28) that

\[
J \leq \int_0^\infty \xi^T(t)W \xi(t) \, dt,
\]

where

\[
\xi^T(t) = \begin{bmatrix} \mathcal{D}^T(z_t) & z^T(t-h) & z^T(t-\tau) & ((1/h) \int_{t-h}^t z(s) \, ds)^T & w^T(t) \end{bmatrix}
\]

and

\[
W = \begin{bmatrix} W_1 & P\Delta A(t) & W_2 & -hP(\Delta A + \Delta A(t))A_1 - hQ A_1 & PB \\
-\Delta A^T & 0 & 0 & 0 & 0 \\
0 & -Q_3 & -Q_5 & -hQ_1 + h^2 Q & 0 \\
0 & 0 & -hQ_1 + h^2 Q & -\gamma^2 I & 0 \\
0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}
\]

\(29\)

where

\[
W_1 = (\Delta A + \Delta A_0(t))^T P + P(\Delta A + \Delta A_0(t)) + Q + L^T L
\]

and

\[
W_2 = P(\Delta A + \Delta A_0(t))A_2 + Q A_2.
\]

\(30\)
Again, using the inequality (23), we have

\[
W \preceq \bar{W} = \begin{bmatrix}
\Omega_1 + \bar{L}^T \bar{L} & 0 & P \bar{A} \bar{A}_2 + \bar{Q} \bar{A}_2 & -h \bar{P} \bar{A} \bar{A}_1 - h \bar{Q} \bar{A}_1 & PB \\
\star & \Omega_2 & 0 & 0 & 0 \\
\star & \star & \Omega_3 & -h \bar{Q} \bar{A}_1 & 0 \\
\star & \star & \star & \Omega_4 & 0 \\
\star & \star & \star & \star & -\gamma^2 I
\end{bmatrix}.
\]

(31)

Here, define the matrices \(Q_1, Q_2, Q_3\) and \(P\) as

\[
Q_1 = (1/h) \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} R_4 & R_5 \\ R_5^T & R_6 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} R_7 & R_8 \\ R_8^T & R_9 \end{bmatrix},
\]

\[
P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},
\]

(32)

where \(P_1, P_2, R_1, R_3, R_4, R_6, R_7\) and \(R_9\) are the positive definite matrix defined in (26).

Utilizing (7) and (32), the inequality \(\bar{W} < 0\) is modified to

\[
\begin{bmatrix}
\Phi_1 & \Phi_2 & 0 & 0 & P_1 A A_2 + (R_1 + R_4 + R_7) A_2 & (R_2 + R_5 + R_8) A_2 \\
\star & \Phi_3 & 0 & 0 & \begin{bmatrix} P_2 (A_0 - K C - G) A_2 \\ + A_2^T (R_2 + R_5 + R_8) \end{bmatrix} & \begin{bmatrix} P_2 (G + A_1) A_2 \\ + (R_3 + R_6 + R_9) A_2 \end{bmatrix} \\
\star & \star & \Omega_8 & \Omega_9 & 0 & 0 \\
\star & \star & \star & \Omega_{10} & 0 & 0 \\
\star & \star & \star & \star & \Omega_{11} & -R_8 \\
\star & \star & \star & \star & \star & -R_9 \\
\star & \star & \star & \star & \star & \star
\end{bmatrix} < 0,
\]

\[
-h P_1 A A_1 - h (R_1 + R_4 + R_7) A_1 \\
\left(-h P_2 (A_0 - K C - G) A_1 \right) \\
\left(-h A_1^T (R_2 + R_5 + R_8)^T \right) \\
0 \\
0 \\
0 \\
-h A_2^T (R_1 + R_4 + R_7) \\
-h (R_2 + R_5 + R_8)^T A_2 \\
\Pi_{12}
\begin{bmatrix}
\Phi_1 & \Phi_2 & 0 & 0 & P_1 B \\
\star & \Phi_3 & 0 & 0 & \begin{bmatrix} P_2 (G + A_1) A_1 \\ - h (R_3 + R_6 + R_9) A_1 \end{bmatrix} \\
\star & \star & \Omega_8 & \Omega_9 & 0 \\
\star & \star & \star & \Omega_{10} & 0 \\
\star & \star & \star & \star & \Omega_{11} & -R_8 \\
\star & \star & \star & \star & \star & -R_9 \\
\star & \star & \star & \star & \star & \star
\end{bmatrix} < 0,
\]

(33)

\[
P_2 B - P_2 K D
\]

\[
< 0,
\]

\[
\gamma^2 I
\]
where
\[ U_1 = P_1 + \varepsilon_0^{-1} P_1 D_0 D_0^T P_1 + \varepsilon_4^{-1} P_1 D_0 D_0^T P_1 + \varepsilon_7^{-1} h P_1 D_0 D_0^T P_1, \]
\[ U_2 = (A_0 - KC - G)^T P_2 + R_2 + R_5 + R_8, \]
\[ U_3 = (G + A_1)^T P_2 + P_2 (G + A_1) + R_3 + R_6 + R_9 + \varepsilon_1^{-1} P_2 D_0 D_0^T P_2 \]
\[ + \varepsilon_2^{-1} P_2 K D_2 D_2 K^T P_2 + \varepsilon_3^{-1} P_2 D_1 D_1 P_2 + \varepsilon_5^{-1} P_2 D_0 D_0^T P_2 \]
\[ + \varepsilon_6^{-1} P_2 K D_2 D_2 K^T P_2 + \varepsilon_8^{-1} h P_2 D_0 D_0^T P_2 + \varepsilon_9^{-1} h P_2 K D_2 D_2 K^T P_2 + L^T L \]
and \( \Pi_1, \Pi_8, \) and \( \Pi_9 \) are defined in (27).

Using some changes of variables, \( M_1 = P_2 G \) and \( M_2 = P_2 K \), the inequality (33) is changed to the LMI (25). Therefore, the inequalities (25) and (26) imply that \( J < 0 \) for any nonzero \( w(t) \in \mathcal{L}_2 \), i.e., the filtering error system has a guaranteed \( \gamma \) level of noise attenuation, which concludes the proof. \( \Box \)

**Remark 2.** Given any solutions of the LMIs (25) and (26) in Theorem 2, the corresponding robust \( \mathcal{H}_\infty \) filter of the form (3) will be constructed from the relations \( G = P_2^{-1} M_1 \) and \( K = P_2^{-1} M_2 \).

**Remark 3.** In this article, in order to solve the LMIs, we utilize Matlab’s LM1 Control Toolbox [15], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [13].

**Example.** To illustrate the proposed method, consider the neural system
\[
\frac{d}{dt} \begin{pmatrix} x(t) - \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} x(t - 1) \end{pmatrix} = \begin{bmatrix} 0 & 3 \\ -4 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} F_0(t) \begin{bmatrix} 0 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0.3 & -0.2 \\ -0.2 & 0.3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} w(t), \]
\[
y(t) = ([1 & 0] + F_1(t) \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}) x(t) + w(t), \]
where \( F_i(t)^T F_i(t) \leq I \) for \( i = 0, 1, 2 \).

The output of error system is defined as
\[ y_\varepsilon(t) = [1 & 1] e(t). \]

The aim is to design a robust \( \mathcal{H}_\infty \) filter (3) for above system. First, checking the stability condition (12) for operator \( \mathcal{D}(z) \) gives the solutions:
Next, by applying the minimization problem of Theorem 2 to above system using LMI toolbox [15], the solutions of optimization problem in the inequalities (25) and (26) are as follows:

\[
P_1 = \begin{bmatrix} 0.1701 & 0.1101 \\ 0.1101 & 0.1615 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.6795 & 0.3221 \\ 0.3221 & 0.9955 \end{bmatrix},
\]

\[
R_1 = \begin{bmatrix} 0.1324 & 0.0867 \\ 0.0867 & 0.1478 \end{bmatrix}, \quad R_2 = \begin{bmatrix} -0.0255 & -0.0144 \\ -0.0210 & -0.0131 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0.4043 & 0.2017 \\ 0.2017 & 0.5774 \end{bmatrix},
\]

\[
R_4 = \begin{bmatrix} 0.0476 & 0.0473 \\ 0.0473 & 0.0485 \end{bmatrix}, \quad R_5 = \begin{bmatrix} -0.0049 & -0.0052 \\ -0.0045 & -0.0051 \end{bmatrix}, \quad R_6 = \begin{bmatrix} 0.1096 & 0.1060 \\ 0.1060 & 0.1115 \end{bmatrix}, \quad R_7 = \begin{bmatrix} 0.1394 & 0.1291 \\ 0.1291 & 0.1539 \end{bmatrix},
\]

\[
R_8 = \begin{bmatrix} -0.0275 & -0.0403 \\ -0.0119 & -0.0463 \end{bmatrix}, \quad R_9 = \begin{bmatrix} 0.8812 & 0.6713 \\ 0.6713 & 0.7702 \end{bmatrix}, \quad M_1 = \begin{bmatrix} -2.5064 & 0.1475 \\ -4.6942 & -4.0887 \end{bmatrix},
\]

\[
M_2 = \begin{bmatrix} 0.9731 \\ 0.7325 \end{bmatrix}, \quad \rho = 0.1812, \quad \varepsilon_0 = 2.3617, \quad \varepsilon_1 = 5.1754, \quad \varepsilon_2 = 4.463,
\]

\[
\varepsilon_3 = 2.0789, \quad \varepsilon_4 = 10^{-4} \times 4.8687, \quad \varepsilon_5 = 29.7619, \quad \varepsilon_6 = 10.8645, \quad \varepsilon_7 = 2.0256, \quad \varepsilon_8 = 4.439,
\]

\[
\varepsilon_9 = 14.5728.
\]

Therefore, in light of Remark 2, the parameters \( G \) and \( K \) of robust \( \mathcal{H}_\infty \) filter are

\[
G = \begin{bmatrix} -1.7168 & 2.5557 \\ -4.1602 & -4.9341 \end{bmatrix}, \quad K = \begin{bmatrix} 1.2795 \\ 0.3219 \end{bmatrix}.
\]

Therefore, the obtained robust \( \mathcal{H}_\infty \) filter stabilizes the filtering error system and minimizes \( \mathcal{H}_\infty \) norm within \( \gamma = \rho^{1/2} = 0.4257 \) in the presence of delay and uncertainties.
4. Conclusions

This paper investigated the design of robust $\mathcal{H}_\infty$ filter for neutral delay-differential system with parametric uncertainties in the system state-space model. A methodology for the design of a stable linear filter that assures asymptotic stability and minimization of $\mathcal{H}_\infty$ performance for the filtering error system is studied. The filter is obtained from a convex optimization problem described in terms of LMIs. The validity of proposed filter design algorithm has been checked through a numerical example.

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