Decentralized dynamic output feedback controller design for guaranteed cost stabilization of large-scale discrete-delay systems

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Abstract

In this paper, we discuss how to solve dynamic output feedback controller design problem for decentralized guaranteed cost stabilization of large-scale discrete-delay system by convex optimization. Based on Lyapunov second method, an linear matrix inequality optimization problem is formulated to design the controller which guarantees the asymptotic stability and minimizes the upper bound of a given quadratic cost function. A numerical example is given to illustrate the proposed method.

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1. Introduction

There exist many real-world systems that can be modelled as large-scale systems: examples are computer communication networks, transportation networks, economic systems, and so on. In general, a large-scale system can be characterized by a large number of variables representing the system, a strong interaction between subsystem variables, and a complex structure [5,11]. Also, time delays are often encountered in large-scale systems because of computation...
data, measurement of system variables, and signal transmission between subsystems. The existence of the delay is frequently a source of instability and poor performance. Therefore, the stabilization problem of large-scale system with time-delay has been one of the most popular research topics in control systems during the last decades (see [4,7,8,12] and reference therein). The majority of these works treated the problem in continuous-time domain. However, since most modern control systems are controlled by a digital computer, it is natural to deal with the problem in discrete-time domain. Furthermore, since the stabilizing controller designed in the literature [4,7,8,12] is based on state-feedback control scheme, their results cannot be applicable when all the states are not available for feedback. In fact, it is a strong requirement to demand the availability of all the states in large-scale systems.

On the other hand, when controlling a real plant, it is also desirable to design a control systems which is not only asymptotically stable but also guarantees an adequate level of performance. One way to address the performance problem is to consider a linear quadratic cost function. This approach is the so-called guaranteed cost control [2,6,9,13]. Up to date, unfortunately, the topic of guaranteed cost control for large-scale systems has been received very little attention.

In this paper, we consider a class of large-scale discrete-time systems with delays in subsystem interconnections. Using the Lyapunov method combined with linear matrix inequality (LMI) technique, we develop a dynamic output feedback controller for guaranteed cost stabilization of the system, which makes the closed-loop system asymptotically stable and guarantees an adequate level of performance. Stabilization criteria for the existence of the guaranteed cost controller are derived in terms of LMIs. The LMI approach has been one of the hot spots of research due to its computational advantage and simplicity in solving the addressed problems. The LMI can be easily solved by various efficient convex optimization algorithms [1].

Notations: Through the paper, $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and $I$ is the identity matrix with appropriate dimensions. $\text{diag}\{\cdots\}$ denotes the block diagonal matrix. $*$ denotes the symmetric part. For $X \in \mathbb{R}^{n \times n}$, the notation $X > 0$ ($X < 0$) means that matrix $X$ is symmetric and positive-definite (negative-definite).

2. Problem formulation

Consider a class of large-scale system composed of $n$ interconnected subsystems described by

$$
x_i(k+1) = A_i x_i(k) + \sum_{j=1, j \neq i}^{n} A_{ij} x_j(k - h_{ij}) + B_i u_i(k),
$$

$$
y_i(k) = C_i x_i(k), \quad i = 1, 2, \ldots, n,
$$

where $A_i$, $B_i$, $C_i$, $h_{ij}$ are matrices and $u_i(k)$ is the input.
where $x_i(k) \in \mathbb{R}^{n_i}$ is the state vector, $u_i(k) \in \mathbb{R}^{m_i}$ is the control vector, $y_i(k) \in \mathbb{R}^{q_i}$ is the output vector, and the time-delays, $h_{ij}$, are the positive constants. The system matrices $A_i, B_i, C_i,$ and $A_{ij}$ are of appropriate dimensions. It is assumed that the triple $(A_i, B_i, C_i), i = 1, \ldots, n,$ is stabilizable and detectable.

In order to stabilize system (1), let’s consider the following dynamic output feedback controller for subsystem $i$:

$$
\xi_i(k + 1) = A_{ci} \xi_i(k) + B_{ci} y_i(k),
$$

$$
\xi_i(0) = 0,
$$

where $\xi_i(k) \in \mathbb{R}^{n_i},$ and $A_{ci}, B_{ci},$ and $C_{ci}$ are constant matrices of appropriate dimensions to be determined later.

The performance index associated with subsystem $i$ is the following quadratic function

$$
J_i = \sum_{k=0}^{\infty} (x_i^T(k)Q_i x_i(k) + u_i^T(k)S_i u_i(k)),
$$

where $Q_i \in \mathbb{R}^{n_i \times n_i}$ and $S_i \in \mathbb{R}^{m_i \times m_i}$ are given positive-definite matrices.

Applying the controller (2) to the system (1) results in the closed-loop system

$$
z_i(k + 1) = \bar{A}_i z_i(k) + \sum_{j=1, j \neq i}^{n} \bar{D}_{ij} z_j(k - h_{ij}),
$$

where for $i = 1, 2, \ldots, n$,

$$
\bar{A}_i = \begin{bmatrix} A_i & B_i C_i \\ B_i C_i A_i & A_i \end{bmatrix}, \quad \bar{D}_{ij} = \begin{bmatrix} A_{ij} & 0 \\ 0 & 0 \end{bmatrix}, \quad z_i(k) = \begin{bmatrix} x_i(k) \\ \xi_i(k) \end{bmatrix}.
$$

The corresponding closed-loop cost function is

$$
J_i = \sum_{k=0}^{\infty} z_i^T(k) \begin{bmatrix} Q_i & 0 \\ 0 & C_i^T S_i C_i \end{bmatrix} z_i(k) = \sum_{k=0}^{\infty} z_i^T(k) \bar{Q}_i z_i(k).
$$

Here, the objective of this paper is to develop a procedure to design a dynamic output feedback controller $u_i(k)$ for system (1) and performance index (3), such that the resulting closed-loop system is asymptotically stable and the closed-loop value of the cost function (3) satisfies $J_i \leq J_i^*$, where $J_i^*$ is some specified constant.

**Definition 1.** For the dynamic system (1) and cost function (3), if there exist a control law $u_i^*(k)$ and a positive constant $J_i^*$ such that for all admissible delays, the closed-loop system (4) is asymptotically stable and the closed-loop value of the cost function (3) satisfies $J_i \leq J_i^*$, then $J_i^*$ is said to be a guaranteed cost and $u_i^*(k)$ is said to be a guaranteed cost control law of subsystem $i$ and its corresponding cost function (3).
Before proceeding further, we will give a well-known fact.

**Fact 2** (Schur complement). Given constant symmetric matrices $\Sigma_1$, $\Sigma_2$, $\Sigma_3$ where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3\Sigma_2^{-1}\Sigma_3 < 0$ if and only if
\[
\begin{bmatrix}
\Sigma_1 & \Sigma_3^T \\
\Sigma_3 & -\Sigma_2
\end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix}
-\Sigma_2 & \Sigma_3 \\
\Sigma_3^T & \Sigma_1
\end{bmatrix} < 0.
\]

3. Controller design

In this section, two criteria for the existence of a dynamic output feedback controller (2) for guaranteed cost stabilization of system (1), will be derived using the Lyapunov theory and LMI convex optimization technique.

The following is a main result of the paper.

**Theorem 3.** For given scalars $h_{ij} > 0$, there exists an dynamic output feedback controller (2) for system (1) if there exist positive-definite matrices $X_i$, $Y_i$, and matrices $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $\hat{W}_i$ satisfying the following LMIs:
\[
-X_i - I \quad X_iA_i + \hat{B}_iC_i \quad 0 \quad 0
\]
\[
-\hat{Y}_i \quad \hat{A}_i \quad \hat{A}_iY_i + B_i\hat{C}_i \quad 0 \quad 0
\]
\[
-\hat{X}_i + (n-1)I + \hat{Q}_i \quad -I + (n-1)Y_i + QY_i \quad 0 \quad 0
\]
\[
-Y_i + (n-1)\hat{W}_i \quad Y_i \quad \hat{C}_i^TS_i \quad \hat{C}_i^TS_i
\]
\[
-I/(n-1) \quad 0
\]
\[
-\hat{S}_i \quad 0
\]
\[
\vdots
\]
\[
X_iA_{i+1} \quad X_iA_{i+2} \quad \cdots \quad X_iA_{i,i-1} \quad X_iA_{i,i+1} \quad \cdots \quad X_iA_{i,n}
\]
\[
A_{i+1} \quad A_{i+2} \quad \cdots \quad A_{i,i-1} \quad A_{i,i+1} \quad \cdots \quad A_{i,n}
\]
\[
0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad \cdots \quad 0
\]
\[
0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad \cdots \quad 0
\]
\[
0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad \cdots \quad 0
\]
\[
0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad \cdots \quad 0
\]
\[
-I \quad 0 \quad \cdots \quad 0 \quad 0 \quad \cdots \quad 0
\]
\[
\ast \quad -I \quad \cdots \quad 0 \quad 0 \quad \cdots \quad 0
\]
\[
\ast \quad \ast \quad \ast \quad \ast \quad \ast \quad \ast \quad \ast \quad \ast \quad \ast
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\[
\ast \quad \ast \quad \ast \quad \ast \quad \ast \quad \ast \quad \ast
\]
\[
\vdots
\]
\[
\ldots
\]
\[
< 0,
\]
\[
(7)
\]
\[
\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0 \text{ for } i = 1, 2, \ldots, n. \tag{8}
\]

Then, the upper bound of cost function for subsystem \(i\) is

\[
J_i \leq x_i^T(0)X_i x_i(0) + \sum_{j=1, j \neq i}^{n} \sum_{l=-h_{ij}}^{-1} x_j^T(l)x_j(l) \triangleq J_i^*. \tag{9}
\]

**Proof.** Consider a Lyapunov function for system (4)

\[
V = \sum_{i=1}^{n} V_i \equiv \sum_{i=1}^{n} \left( z_i^T(k)P_i z_i(k) + \sum_{j=1, j \neq i}^{n} \sum_{l=k-h_{ij}}^{k-1} z_j^T(l)z_j(l) \right),
\]

where \(P_i > 0\) for \(i = 1, 2, \ldots, n\).

By simple calculation, the corresponding Lyapunov difference \((\Delta V = V(k+1) - V(k))\) is found as

\[
\Delta V = \sum_{i=1}^{n} \left[ z_i^T(k) \left( \bar{A}_i^T P_i \bar{A}_i - P_i \right) z_i(k) + 2z_i^T(k) \bar{A}_i^T P_i \sum_{j=1, j \neq i}^{n} \bar{D}_{ij} z_j(k-h_{ij}) \\
+ \left( \sum_{j=1, j \neq i}^{n} \bar{D}_{ij} z_j(k-h_{ij}) \right)^T P_i \left( \sum_{j=1, j \neq i}^{n} \bar{D}_{ij} z_j(k-h_{ij}) \right) \\
- \left( \sum_{j=1, j \neq i}^{n} z_j(k-h_{ij}) \right)^T \left( \sum_{j=1, j \neq i}^{n} z_j(k-h_{ij}) \right) + \sum_{j=1, j \neq i}^{n} z_j^T(k)z_j(k) \right]. \tag{10}
\]

Note that

\[
\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} z_j^T(k)z_j(k) = (n-1) \sum_{i=1}^{n} z_i^T(k)z_i(k). \tag{11}
\]
Using the relation (11), it follows that
\[
\Delta V = \sum_{i=1}^{n} Z_i^T \begin{bmatrix}
(1, 1) & \tilde{A}_i^T P_i \tilde{D}_{i1} & \ldots & \tilde{A}_i^T P_i \tilde{D}_{in} \\
(2, 2) & \tilde{D}_{i1}^T P_i \tilde{D}_{i2} & \ldots & \tilde{D}_{i1}^T P_i \tilde{D}_{in} \\
& \ddots & \ddots & \ddots \\
& & * & * & \ldots & (n, n)
\end{bmatrix} Z_i
\]
\[-\sum_{i=1}^{n} z_i^T(k) \bar{Q}_i z_i(k)
\equiv \sum_{i=1}^{n} Z_i^T M_i Z_i - \sum_{i=1}^{n} z_i^T(k) \bar{Q}_i z_i(k),
\]
where \((1, 1) = \tilde{A}_i^T P_i \tilde{A}_i - P_i + (n - 1)I + \bar{Q}_i,\) \((2, 2) = \tilde{D}_{i1}^T P_i \tilde{D}_{i1} - I,\) \((3, 3) = \tilde{D}_{i2}^T P_i \tilde{D}_{i2} - I,\) \((n, n) = \tilde{D}_{in}^T P_i \tilde{D}_{in} - I,\) \(Z_i = [z_i^T(k) \ z_i^T(k - h_{i1}) \ z_i^T(k - h_{i2}) \ \ldots \ z_i^T(k - h_{i,i-1}) \ z_i^T(k - h_{i,i+1}) \ \ldots \ z_i^T(k - h_{in})].\)

Therefore, if \(M_i < 0\) for all \(i,\) there exist the positive scalars \(\gamma_i\) such that
\[
\Delta V \leq -\sum_{i=1}^{n} z_i^T(k) \bar{Q}_i z_i(k) \leq -\sum_{i=1}^{n} \gamma_i \|x_i(k)\|^2
\]
which guarantees the asymptotic stability of the system by Lyapunov stability theory.

By Fact 2, the inequality, \(M_i < 0\ \forall i,\) is equivalent to the following inequality:
\[
M_i \equiv \begin{bmatrix}
-P_i^{-1} & \tilde{A}_i & \tilde{D}_i \\
* & -P_i + (n - 1)I + \bar{Q}_i & 0 \\
* & * & -\bar{I}_i
\end{bmatrix} < 0 \quad \text{for } i = 1, 2, \ldots, n,
\]
(13)
where \(\bar{D}_i = [\tilde{D}_{i1}, \tilde{D}_{i2}, \ldots, \tilde{D}_{i,i-1}, \tilde{D}_{i,i+1}, \ldots, \tilde{D}_{in}]\) and \(\bar{I}_i = \text{diag}\{I_1, I_2, \ldots, I_{i-1}, I_{i+1}, \ldots, I_n\}.\) Here, \(I_j\) denotes the identity matrix of \(j\)th diagonal entry in the matrix \(\bar{I}_i.\)

Postmultiplying and premultiplying the matrix inequality (13) by the matrix \(\text{diag}\{I, P_i^{-1}, I\}\) and by its transpose, respectively, gives
\[
\tilde{M}_i \equiv \begin{bmatrix}
-V_i & \tilde{A}_i V_i & \tilde{D}_i \\
* & -V_i + (n - 1)V_i V_i + V_i \bar{Q}_i V_i & 0 \\
* & * & -\bar{I}_i
\end{bmatrix} < 0 \quad \text{for } i = 1, 2, \ldots, n,
\]
(14)
where \(V_i = P_i^{-1}.\)
In the matrix $\hat{M}_i$, the matrices $V_i > 0$ and the controller parameters $A_{ci}$, $B_{ci}$ and $C_{ci}$, which included in the matrix $\hat{A}_i$, are unknown and occur in nonlinear fashion. Hence, the inequality $\hat{M}_i < 0$ cannot be considered as an LMI convex optimization problem. In the following, we will use a method of changing variables such that the inequality can be solved as convex optimization algorithms [10].

First, partition the matrix $V_i$ and its inverse as

$$V_i = \begin{bmatrix} Y_i & N_i \\ N_i^T & W_i \end{bmatrix}, \quad V_i^{-1} = \begin{bmatrix} X_i & M_i \\ M_i^T & U_i \end{bmatrix},$$

where $X_i, Y_i \in \mathbb{R}^{n \times n}$. Note that the equality $V_i V_i^{-1} = I$ gives that

$$M_i N_i^T = I - X_i Y_i. \quad (15)$$

Now, define

$$F_i = \begin{bmatrix} X_i & I \\ M_i^T & 0 \end{bmatrix}, \quad G_i = \begin{bmatrix} I & Y_i \\ 0 & N_i^T \end{bmatrix}.$$

Then, it follows that

$$V_i F_i = G_i, \quad F_i^T V_i F_i = G_i^T F_i = \begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0. \quad (16)$$

Postmultiplying and premultiplying the matrix inequality (14) by the matrix $\text{diag}\{F_i, F_i, I\}$ and by its transpose, respectively, gives

$$\begin{bmatrix} -F_i^T V_i F_i & F_i^T \hat{A}_i V_i F_i \\ * & -F_i^T V_i F_i + (n - 1) F_i^T V_i V_i F_i + F_i^T V_i \hat{Q}_i V_i F_i & 0 \\ * & * & -I_i \end{bmatrix} < 0. \quad (17)$$

By using matrix operations, it is straightforward to verify that

$$F_i^T V_i \hat{Q}_i V_i F_i = \begin{bmatrix} Q_i & * \\ * & N_i C_{ci} S_i C_{ci} N_i^T \end{bmatrix},$$

$$F_i^T \hat{A}_i V_i F_i = \begin{bmatrix} X_i A_i + M_i B_{ci} C_i & X_i A_i Y_i + M_i B_{ci} C_i Y_i + X_i B_i C_{ci} N_i^T + M_i A_{ci} N_i^T \\ A_i & A_i Y_i + B_i C_{ci} N_i^T \end{bmatrix},$$

$$F_i^T \hat{D}_i = \begin{bmatrix} X_i A_{i1} & 0 & \cdots & 0 & X_i A_{i,i-1} & 0 & \cdots & X_i A_{i,i+1} & 0 & \cdots & X_i A_{in} \\ A_{i1} & A_{i2} & \cdots & A_{i,i-1} & A_{i,i+1} & \cdots & A_{in} & 0 \end{bmatrix},$$

$$F_i^T V_i V_i F_i = \begin{bmatrix} I & * \\ * & Y_i Y_i + N_i N_i^T \end{bmatrix}. \quad (18)$$
Now, define a new set of variables as follows:

\[
\tilde{W}_i = N_i N_i^T, \\
\tilde{A}_i = X_i A_i Y_i + X_i B_i \tilde{C}_i + \tilde{B}_i C_i Y_i + M_i A_i e_i e_i^T N_i^T, \\
\tilde{B}_i = M_i B_i, \\
\tilde{C}_i = C_i e_i e_i^T N_i^T.
\]

(19)

By using (18) and (19), the inequality (17) is modified as

\[
\begin{bmatrix}
-X_i & -I & X_i A_i + \tilde{B}_i C_i & \tilde{A}_i \\
* & -Y_i & A_i & A_i Y_i + B_i \tilde{C}_i \\
* & * & -X_i + (n-1)I + Q_i & -I + (n-1)Y_i + Q_i Y_i \\
* & * & * & -Y_i + (n-1)Y_i Y_i + (n-1)\tilde{W}_i + \tilde{C}_i^T S_i \tilde{C}_i \\
* & * & * & * \\
\vdots & & \vdots & \vdots \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\end{bmatrix} < 0, \quad \forall i.
\]

(20)
By Fact 2 (Schur complement), it follows that the above inequality (20) is equivalent to the matrix inequality (7).

Since the matrix inequality (7) implies that
\[ \Delta V_i \leq -z_i^T(k)\overline{Q}_i z_i(k) < 0, \] (21)
we have
\[ z_i^T(k)\overline{Q}_i z_i(k) < -\Delta V_i = V_i(k) - V_i(k + 1). \]

Summing both sides of the above inequality from 0 to \( \infty \) leads to
\[ J_i = \sum_{k=0}^{\infty} z_i^T(k)\overline{Q}_i z_i(k) \leq V_i(0) - V_i(\infty). \]

Since the asymptotic stability of the system has already been established, we conclude that \( V_i(k) \to 0 \) as \( k \to \infty \). Hence, we have
\[ J_i \leq z_i^T(0)P_i z_i(0) + \sum_{j=1}^{n} \sum_{l=-h_{ij}}^{-1} z_j^T(l)z_j(l) \]
\[ = x_i^T(0)X_i x_i(0) + \sum_{j=1}^{n} \sum_{l=-h_{ij}}^{-1} x_j^T(l)x_j(l) = J_i^*. \] (22)

This completes the proof. \( \square \)

Theorem 3 presents a method of designing a dynamic output feedback guaranteed cost controller. The following theorem presents a method of selecting a controller minimizing the upper bound of the guaranteed cost (9).

**Theorem 4.** Consider the system (1) with cost function (3). For all \( i \), if the following optimization problem

\[
\min_{\bar{z}_i, \bar{X}_i, \bar{Y}_i, \bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{W}_i} \bar{z}_i
\]
subject to

(i) LMIIs (7) and (8),

(ii) \[
\begin{bmatrix}
-\bar{z}_i & X_i^T(0)X_i \\
* & -X_i
\end{bmatrix} < 0,
\]

has a solution set \((\bar{z}_i, \bar{X}_i, \bar{Y}_i, \bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{W}_i)\), the controller (2) is an optimal decentralized dynamic output feedback guaranteed cost controller which ensures the minimization of the guaranteed cost (9) for large-scale discrete-time system (1). Then the optimal cost is \( J_i^* = \bar{z}_i + \Phi_i \), where \( \Phi_i \triangleq \sum_{j=1, j \neq i}^{n} \sum_{l=-h_{ij}}^{-1} x_j^T(l)x_j(l) \).

**Proof.** By Theorem 3, (i) in (23) is clear. Also, it follows from the Fact 2 that (ii) in (23) is equivalent to \( x_i^T(0)X_i x_i(0) < \bar{z}_i \). So, it follows from (22) that
Thus, the minimization of $z_i$ implies the minimization of the guaranteed cost for the system (1). The convexity of this optimization problem ensures that a global optimum, when it exists, is reachable.

**Remark 5.** The problem of Theorem 3 and 4 is to determine whether the problem is feasible or not. It is called the feasibility problem. The solutions of the problem can be found by solving generalized eigenvalue problem in $X_i$, $Y_i$, $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $\hat{W}_i$, which is a convex optimization problem. Various efficient convex optimization algorithms can be used to check whether the LMIs are feasible. In this paper, in order to solve the matrix inequality, we utilize Matlab’s LMI Control Toolbox [3], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [1].

**Remark 6.** Given any solution of the LMIs (23) in Theorem 4, a corresponding controller of the form (2) for subsystem $i$ will be constructed as follows:

- Using two solutions $X_i$, $Y_i$, compute the invertible matrices $N_i$ satisfying the relation $\hat{W}_i = N_iN_i^T$.
- Using the matrix $N_i$, compute the invertible matrix $M_i$ satisfying (16).
- Utilizing the matrices $M_i$ and $N_i$ obtained above, solve the system of equations (19) for $B_{ci}$, $C_{ci}$ and $A_{ci}$ (in this order).

**Remark 7.** Uncertainties in system (1) are not considered for simplicity. However, the results obtained can be easily generalized to system with uncertainties.

To illustrate the application of the proposed method, we present the following example.

**Example 8.** Consider a large-scale system which is composed of the following two interconnected subsystems

$$
J_i^* = z_i + \Phi_i.
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Thus, the minimization of $z_i$ implies the minimization of the guaranteed cost for the system (1). The convexity of this optimization problem ensures that a global optimum, when it exists, is reachable.

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**Remark 6.** Given any solution of the LMIs (23) in Theorem 4, a corresponding controller of the form (2) for subsystem $i$ will be constructed as follows:

- Using two solutions $X_i$, $Y_i$, compute the invertible matrices $N_i$ satisfying the relation $\hat{W}_i = N_iN_i^T$.
- Using the matrix $N_i$, compute the invertible matrix $M_i$ satisfying (16).
- Utilizing the matrices $M_i$ and $N_i$ obtained above, solve the system of equations (19) for $B_{ci}$, $C_{ci}$ and $A_{ci}$ (in this order).

**Remark 7.** Uncertainties in system (1) are not considered for simplicity. However, the results obtained can be easily generalized to system with uncertainties.

To illustrate the application of the proposed method, we present the following example.

**Example 8.** Consider a large-scale system which is composed of the following two interconnected subsystems

$$
x_1(k+1) = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1.5 \end{bmatrix}x_1(k) + \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}x_2(k-h_{12}) + \begin{bmatrix} 2 \\ 5 \end{bmatrix}u_1(k),
$$

$$
y_1(k) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}x_1(k),
$$

$$
x_2(k+1) = \begin{bmatrix} 1.2 & 1 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -1 \end{bmatrix}x_2(k) + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}x_1(k-h_{21})
$$

$$
+ \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}u_2(k),
$$

$$
J_i^* = z_i + \Phi_i.
$$

Thus, the minimization of $z_i$ implies the minimization of the guaranteed cost for the system (1). The convexity of this optimization problem ensures that a global optimum, when it exists, is reachable.

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$$

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$$

$$
+ \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}u_2(k),
$$

$$
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$$

Thus, the minimization of $z_i$ implies the minimization of the guaranteed cost for the system (1). The convexity of this optimization problem ensures that a global optimum, when it exists, is reachable.
\[ y_2(k) = [0.1 \ 0 \ 0.1]x_2(k), \]

where \( h_{12} = 3, \ h_{21} = 5 \) and the initial conditions of each subsystem are as follows:

\[ x_1(k) = [1 \ -1]^T, \ x_2(k) = [0.5 \ 2 \ -2]^T \text{ for } -h_{21} \leq k \leq 0. \]

Associated with this system is the cost function of (3) with \( Q_1 = I, \ Q_2 = I, \ S_1 = I \) and \( S_2 = I \). Here, by solving the optimization problem (23) of Theorem 4 for subsystem 1 and 2, we find the positive solutions as

\[
X_1 = \begin{bmatrix} 8.1578 & 6.3427 \\ 6.3427 & 11.6876 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 0.7461 & 0.1762 \\ 0.1762 & 0.5754 \end{bmatrix}, \\
\hat{A}_1 = \begin{bmatrix} -0.3451 & -0.3149 \\ -0.1705 & 0.0857 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} -39.9087 \\ -73.5786 \end{bmatrix}, \\
\hat{C}_1 = \begin{bmatrix} -0.0998 & -0.2126 \end{bmatrix}, \quad \hat{W}_1 = 10^{-5} \times \begin{bmatrix} 0.6500 & 0.3285 \\ 0.3285 & 0.4120 \end{bmatrix},
\]

\[ x_1 = 7.1601. \]

\[
X_2 = \begin{bmatrix} 6.4385 & 2.9798 & -2.8126 \\ 2.9798 & 9.0664 & 0.2124 \\ -2.8126 & 0.2124 & 7.3899 \end{bmatrix}, \\
Y_2 = \begin{bmatrix} 0.6144 & -0.0455 & -0.1191 \\ -0.0455 & 0.6410 & 0.0905 \\ -0.1191 & 0.0905 & 0.7196 \end{bmatrix}, \\
\hat{A}_2 = \begin{bmatrix} 0.0332 & -0.2817 & 0.1715 \\ -0.0130 & -0.1786 & 0.5825 \\ -0.3114 & 0.4614 & 0.1612 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} -42.5097 \\ -20.8794 \\ 41.6483 \end{bmatrix}, \\
\hat{C}_2 = \begin{bmatrix} -0.3845 & -0.2580 & 0.0097 \\ 0.1008 & 0.1058 & 0.2557 \end{bmatrix}, \\
\hat{W}_2 = 10^{-3} \times \begin{bmatrix} 0.0298 & 0.0313 & 0.0632 \\ 0.0313 & 0.1386 & 0.1295 \\ 0.0632 & 0.1295 & 0.1751 \end{bmatrix},
\]

\[ x_2 = 77.3202. \]

Now, according to the procedure of Remark 6, we can obtain the gain matrices of controller of each subsystem:
and the optimal guaranteed costs of closed-loop system of each subsystem are as follows:

\[ J_1^* = \alpha_1 + \Phi_1 = 31.9101, \]

\[ J_2^* = \alpha_2 + \Phi_2 = 87.3202. \]

The numerical simulation results are in Figs. 1 and 2. In the figures, one can see that the system is indeed well stabilized irrespective of time-delays via dynamic output guaranteed cost controller designed.

![Fig. 1. State responses of subsystem 1.](image-url)
Acknowledgements

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References