LMI optimization approach to asymptotic stability of certain neutral delay differential equation with time-varying coefficients

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Abstract

In this article, a stability criterion for all solutions of a class of neutral equation of the form
\[
\frac{d}{dt} [x(t) + c(t)x(t - \tau)] + p(t)x(t) + q(t)x(t - \sigma) + r(t) \int_{t-\delta}^{t} x(s) \, ds = 0
\]
to approach zero at \( t \to \infty \) is presented. Using the Lyapunov method, the condition, which is expressed in terms of linear matrix inequality, is delay-dependent and can be easily solved by various efficient convex optimization algorithm.

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1. Introduction

The main purpose of this article is to investigate the asymptotic behaviors of solutions of the neutral delay equation
\[
\frac{d}{dt}[x(t) + c(t)x(t - \tau)] = -p(t)x(t) - q(t)x(t - \sigma) - r(t) \int_{t-\delta}^{t} x(s) \, ds, \quad t \geq t_0, 
\]

(1)

where \(c(t), p(t), q(t), r(t) : [t_0, \infty) \rightarrow [0, \infty)\) are continuous, and \(c(t)\) is differentiable with locally bounded derivative, \(\tau, \delta\) and \(\sigma\) are positive real numbers, and \(H = \max\{\tau, \delta, \sigma\}\). With each solution \(x(t)\) of Eq. (1) we assume the initial condition:

\[
x(s) = \phi(s), \quad s \in [t_0 - H, t_0] \quad \text{where} \quad \phi \in \mathcal{C}([t_0 - H, t_0], \mathbb{R}).
\]

Neutral delay differential equations/systems of various types including Eq. (1) have been investigated by many authors (see [1–5,7], and references cited therein).

Recently, the asymptotic stability of Eq. (1) with \(r(t) = 0\) has been discussed in [6], and several sufficient conditions for stability have been presented.

In this article, we shall establish a delay-dependent stability criterion for all solutions of Eq. (1) to approach zero as \(t \to \infty\) using the Lyapunov functional method and linear matrix inequality technique.

Through the article, \(*\) represents the elements below the main diagonal of a symmetric matrix. The notation \(X > Y\), where \(X\) and \(Y\) are matrices of same dimensions, means that the matrix \(X - Y\) is positive definite.

2. Main result

We assume that there exist nonnegative constants \(c_1, p_1, p_2, q_1, q_2, r_1,\) and \(r_2\) such that for \(t \geq t_0,\)

\[
p_1 \leq p(t) \leq p_2, \quad q_1 \leq q(t) \leq q_2, \quad r_1 \leq r(t) \leq r_2,
\]

\[
c(t) \leq c_1 < 1, \quad |c'(t)| \leq c_2.
\]

(2)

Here, Eq. (1) can be written in the following form:

\[
\frac{d}{dt}\left(x(t) + c(t)x(t - \tau) - r(t) \int_{t-\delta}^{t} (\delta - t + s)x(s) \, ds\right) = -(p(t) + \delta r(t))x(t) - q(t)x(t - \sigma), \quad t \geq t_0.
\]

(3)

Define the operator \(\mathcal{D}(x_t) : \mathcal{C} \rightarrow \mathcal{R}\) as

\[
\mathcal{D}(x_t) = x(t) + c(t)x(t - \tau) - r(t) \int_{t-\delta}^{t} (\delta - t + s)x(s) \, ds.
\]

(4)

**Lemma 1.** A sufficient condition for stability of the operator \(\mathcal{D}(x_t)\) is

\[
c_1 + (1/2)\delta^2 r_2 < 1.
\]

(5)
Proof. The proof is obvious and omitted. □

Now, we have the following theorem.

**Theorem 2.** For given \( \sigma > 0 \) and \( \delta > 0 \), every solution \( x(t) \) of Eq. (1) satisfies \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \), if the operator \( \mathcal{D}(x_t) \) is stable and there exist the positive scalars \( \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \alpha, \beta \) and \( \rho \) such that the linear matrix inequality (LMI) holds

\[
\Sigma(x, \beta, \rho, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \begin{bmatrix}
\varepsilon_0 & \sqrt{\mu \varepsilon_1} & \sqrt{\mu \varepsilon_2} & \sqrt{r_2 \mu \delta} & 0 & 0 & 0 & 0 & 0 \\
* & -\varepsilon_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -2\varepsilon_4 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \varepsilon_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \varepsilon_3 & 0 & 0 & 0 \\
* & * & * & * & * & * & \varepsilon_4 & 0 & 0 \\
* & * & * & * & * & * & * & \Sigma_{4,4} & 0
\end{bmatrix} < 0,
\]

where \( \mu = p_2 + \delta r_2 \), \( \Sigma_{1,1} = -2(p_1 + \delta r_1) + \alpha + \beta + 0.5 \delta^2 \rho \), \( \Sigma_{2,2} = q_2 \varepsilon_1 + q_2 c_1 \varepsilon_2 + \beta \), \( \Sigma_{3,3} = \mu c_1 \varepsilon_0 - \alpha \), and \( \Sigma_{4,4} = r_2 q_2 \varepsilon_3 + r_2 \mu \varepsilon_4 - \rho \).

Proof. Consider the legitimate Lyapunov functional [1] defined by

\[
V(t) = \mathcal{D}^2(x_t) + \alpha \int_{t-\tau}^{t} x^2(s) \, ds + \beta \int_{t-\sigma}^{t} x^2(s) \, ds + (1/2) \rho \int_{t-\delta}^{t} (\delta - t + s)^2 x^2(s) \, ds,
\]

where \( \alpha, \beta, \) and \( \rho \) are positive scalars to be determined later.

The derivative of \( V(t) \) along the solution of Eq. (3) is given by

\[
\frac{dV}{dt} = -2 \left( x(t) + c(t)x(t-\tau) - r(t) \int_{t-\delta}^{t} (\delta - t + s)x(s) \, ds \right) \\
\quad \left[ (p(t) + \delta r(t))x(t) + q(t)x(t-\sigma) \right] + \alpha x^2(t) - \beta x^2(t-\tau) \\
+ (1/2) \delta^2 p x^2(t) - \rho \int_{t-\delta}^{t} (\delta - t + s)x^2(s) \, ds + \beta x^2(t) - \beta x^2(t-\sigma),
\]
Using the well-known inequality $2uv \leq \varepsilon u^2 + \nu^{-1}v^2$ for $\varepsilon > 0$, we have

$$
\frac{dV}{dt} \leq \left[ -2(p_1 + \delta r_1) + \alpha + \beta + 0.5\delta^2 \rho \right] x^2(t) + \mu_c \left[ \varepsilon_0^{-1} x^2(t) + \varepsilon_0 x^2(t - \tau) \right] + q_2 \left[ \varepsilon_0^{-1} x^2(t) + \varepsilon_1 x^2(t - \sigma) \right] + q_2 \varepsilon_1 \left[ \varepsilon_2^{-1} x^2(t - \tau) + \varepsilon_2 x^2(t - \sigma) \right] + r_2 q_2 \left[ 0.5\varepsilon_0^{-1} \delta^2 x^2(t - \sigma) + e_3 \int_{\tau-t}^{\tau-t} (\delta - t + s)x^2(s) \, ds \right] + r_2 \mu \left[ 0.5\varepsilon_1^{-1} \delta^2 x^2(t) + e_4 \int_{\tau-t}^{\tau-t} (\delta - t + s)x^2(s) \, ds \right] - \rho \int_{\tau-t}^{\tau-t} (\delta - t + s)x^2(s) \, ds - \alpha x^2(t - \tau) - \beta x^2(t - \sigma),
$$

where $\mu$ is defined in Theorem 2.

Thus, we have a new bound of $\frac{dV}{dt}$ in the form of

$$
\frac{dV}{dt} \leq \chi^T(t) \begin{bmatrix}
(1, 1) & 0 & 0 & 0 \\
* & (2, 2) & 0 & 0 \\
* & * & (3, 3) & 0 \\
* & * & * & (4, 4)
\end{bmatrix} \chi(t) \equiv \chi^T(t) \Omega \chi(t),
$$

(10)
where \((1, 1) = -2(p_1 + \delta r_1) + \alpha + \beta + 0.5 \delta^2 \rho + \mu c_1 e_0^{-1} + q_2 e_1^{-1} + 0.5 r_2 \mu e_4^{-1} \delta^2,\)
\((2, 2) = q_2 e_1 + q_2 c_1 e_2 - \beta + 0.5 r_2 q_2 e_3^{-1} \delta^2,\)
\((3, 3) = q_2 e_1 e_2^{-1} + \mu c_1 e_0 - \alpha,\)
\((4, 4) = r_2 q_2 e_3 + r_2 \mu e_4 - \rho,\)
and
\[
\chi(t) = \begin{bmatrix}
x(t) \\
x(t - \sigma) \\
x(t - \tau) \\
\int_{t-\delta}^{t} (\delta - t + s)x^2(s) \, ds
\end{bmatrix}.
\]

Therefore if the matrix \(\Omega\) is negative definite, \(\frac{dV}{dt}\) is negative. By Schur complement [8], the fact that \(\Omega < 0\) is equivalent to \(\Sigma(\cdot) < 0\). Eq. (6) implies that \(\frac{dV}{dt} \leq -\gamma \|x(t)\|^2\) for sufficiently small \(\gamma > 0\). Noting that the operator \(\mathcal{D}(x_t)\) is stable, therefore, Eq. (1) is asymptotically stable according to [1, Theorem 8.1, pp. 292–293]. This completes the proof. \(\Box\)

**Remark 3.** The problem (6) is to determine whether the problem is feasible or not. It is called the feasibility problem. The solutions of the problem can be found by solving eigenvalue problem with respect to \(\alpha, \beta, \rho, e_0, e_1, e_2, e_3\) and \(e_4\), which is a convex optimization problem [8]. Various efficient convex optimization algorithms can be used to check whether the inequality (6) is feasible. In the article, in order to solve the linear matrix inequality (6), we utilize Matlab’s LMI control toolbox [9], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [8]. Therefore, all solutions \((\alpha, \beta, \rho, e_0, e_1, e_2, e_3, e_4)\) can be obtained at the same time.

**Remark 4.** In the work [6], the two sufficient condition for stability of the system (1) with \(r(t) = 0\) have been presented as

\[
p_1 + q_1 > (p_2 + q_2)(c_1 + q_2),
\]
\[(11)\]
\[
p_1 > q_2 + c_1(p_2 + q_2).
\]
\[(12)\]

Eq. (1) with \(r(t) = 0\) can be written in the following form:

\[
\frac{d}{dt} \left( x(t) + c(t)x(t - \tau) - q(t) \int_{t-\sigma}^{t} x(s) \, ds \right) = -(p(t) + q(t))x(t), \quad t \geq t_0.
\]
\[(13)\]

**Theorem 5.** For given \(\sigma > 0\), every solution \(x(t)\) of Eq. (13) satisfies \(x(t) \to 0\) as \(t \to \infty\), if \(c_1 + \sigma q_2 < 1\) and there exist the positive scalars \(e_1, e_2, \alpha\) and \(\beta\) such that the LMI satisfies
\[
\begin{bmatrix}
-2(p_1 + q_1) + c_1(p_2 + q_2) + \alpha + \beta \sigma & q_2 & p_2 & 0 & 0 \\
* & -\varepsilon_1 & 0 & 0 & 0 \\
* & * & -\varepsilon_2 & 0 & 0 \\
* & * & * & c_1(p_2 + q_2) - \alpha & 0 \\
* & * & * & * & \varepsilon_1 \sigma_2^2 + \varepsilon_2 \sigma_2^2 - \beta \sigma
\end{bmatrix} < 0.
\]

(14)

**Proof.** From the Lyapunov functional defined by

\[
V(t) = \mathcal{D}^2(x_t) + \alpha \int_{t-\tau}^t x^2(s) ds + \beta \int_{t-\sigma}^t (\sigma - t + s)x^2(s) ds,
\]

\[
\alpha > 0, \quad \beta > 0,
\]

the condition (14) can be easily obtained by similar procedure as in Theorem 2, so the detail proof is omitted. \qed

**Remark 6.** Note that the criteria (6) and (14) is independent of delay \(\tau\).

To demonstrate the applications of main result, we give the following example.

**Example 7.** Consider Eq. (1) with

\[
p_1 = p_2 = 1, \quad q_1 = 0.3, \quad r_1 = r_2 = 0.2, \quad c_1 = 0.25, \quad \tau = 0.25,
\]

\[
\delta = 0.25, \quad \sigma = 0.5.
\]

Then, we want to find the maximum allowable bound of \(q_2\) for asymptotic stability of the equation by Theorem 2. By solving the LMI (6) iteratively with respect to \(q_2\), we found that the upper bound of \(q_2\) is 0.6203. For \(q_2 = 0.6203\), the positive solutions of the LMI are as follows:

\[
\alpha = 0.4176, \quad \beta = 0.7809, \quad \varepsilon_0 = 1.000, \quad \varepsilon_1 = 1.000, \quad \varepsilon_2 = 1.0001,
\]

\[
\varepsilon_3 = 1.4102, \quad \varepsilon_4 = 0.9985, \quad \rho = 0.3848.
\]

**Example 8.** Consider the following equation studied in [6]:

\[
\frac{dx}{dt} \left( x(t) + \frac{1}{t} x(t - \sigma) \right) = -x(t) - \left( \frac{1}{t} + \frac{1}{t^2} \right) x(t - \sigma), \quad t \geq 2.
\]

Actually, the system has a solution \(x(t) = e^{-t} \to 0\) as \(t \to \infty\). Let us take \(c_1 = 0.5, \ c_2 = 0.25, \ p_1 = p_2 = 1, \ q_1 = 0, \) and \(q_2 = 0.75\). Then by solving the linear matrix inequality (14), we found that the upper bound of \(\sigma\) for stability is \(\sigma < 2/21\), which is same result derived in [6].
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References