A new stability analysis of delayed cellular neural networks

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Abstract

In this paper, the global asymptotic stability of delayed cellular neural networks (DCNNs) is investigated. A novel criterion for the stability using the Lyapunov stability theory and linear matrix inequality (LMI) framework is presented. The criterion expressed by LMIs is delay-dependent. The result is less conservative than those established in the earlier references. A numerical example is given to show the effectiveness of proposed method.

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1. Introduction

Since cellular neural networks (CNNs) have been introduced by Chua and Yang [1], various aspects of different neural networks such as Hopfield neural networks, cellular neural networks, Lotka–Volterra neural networks, and bidirectional associative memory neural networks have received a great deal of interest due to their extensive applications in the fields of signal processing, pattern recognition, fixed-point computation, optimization and associative memories, and so on (see [1–5] and the references cited therein). However, in order to deal with moving images, one must introduce the time delays in the signal transmission among the cells. This leads to the model of delayed cellular neural networks (DCNNs). Thus the stability analysis of DCNNs has become an important topic of theoretical studies in neural networks [6–13].

In this paper, we deal with the problem of global asymptotic stability for DCNNs. By constructing a suitable Lyapunov functional, a new condition for global asymptotic stability of DCNNs with constant delay is given in terms of LMIs. The advantage of the proposed approach is that resulting stability criterion can be used efficiently via existing numerical convex optimization algorithms such as the interior-point algorithms for solving LMIs [14].

Notation: \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space, and \( \mathbb{R}^{n \times m} \) is the set of all \( n \times m \) real matrices. \( I \) denotes the identity matrix with appropriate dimensions. \( \| \cdot \| \) denotes the Euclidean norm of given vector.
\(\star\) denotes the elements below the main diagonal of a symmetric block matrix. For symmetric matrices \(X\) and \(Y\), the notation \(X > Y\) (respectively, \(X \geq Y\)) means that the matrix \(X - Y\) is positive definite, (respectively, nonnegative). \(\text{diag}\{\cdots\}\) denotes the block diagonal matrix. \(\mathcal{C}_{n,h} = \mathcal{C}([-h,0], \mathbb{R}^n)\) denotes the Banach space of continuous functions mapping the interval \([-h,0]\) into \(\mathbb{R}^n\), with the topology of uniform convergence.

2. Problem statement

The delayed cellular neural network model to be considered herein is described by

\[
\begin{align*}
\dot{y}_i(t) &= -a_i y_i(t) + \sum_{j=1}^n w_{ij} f_j(y_j(t)) + \sum_{j=1}^n w_{ij}^T f_j(y_j(t-h)) + b_i, \quad i = 1, 2, \ldots, n, \\
\dot{y}(t) &= -Ay(t) + Wf(y(t)) + W_1 f(y(t-h)) + b,
\end{align*}
\]

or equivalently

\[
\begin{align*}
\dot{y}(t) &= -Ay(t) + Wf(y(t)) + W_1 f(y(t-h)) + b,
\end{align*}
\]

where \(y(t) = [y_1(t), \ldots, y_n(t)]^T \in \mathbb{R}^n\) is the neuron state vector, \(f(y(t)) = [f_1(y_1(t)), \ldots, f_n(y_n(t))]^T \in \mathbb{R}^n\) is the activation functions, \(f_j(y(t-h)) = [f_1(y_1(t-h)), \ldots, f_n(y_n(t-h))]^T \in \mathbb{R}^n\), \(b = [b_1, \ldots, b_n]^T\) is a constant input vector, \(h > 0\) is any time delay, \(A = \text{diag}(a_i)\) is a positive diagonal matrix, and \(W = (w_{ij})_{n \times n}\) and \(W_1 = (w_{ij}^0)_{n \times n}\) are the interconnection matrices representing the weight coefficients of the neurons.

Since \(f_j\), \(i = 1, 2, \ldots, n\) are bounded, one can easily prove that there exists at least one equilibrium point for system (2) by using the well-known Brouwer's fixed point theorem [6].

Assume that \(y^* = (y_1^*, y_2^*, \ldots, y_n^*)^T\) is an equilibrium point of the Eq. (2), then we will shift the equilibrium point \(y^*\) to the origin. The transformation \(x(t) = y(t) - y^*\) puts system (2) into the following form

\[
\begin{align*}
\dot{x}(t) &= -Ax(t) + Wg(x(t)) + W_1 g(x(t-h)),
\end{align*}
\]

where \(x(t)\) is the state vector of the transformed system, \(g(x) = [g_1(x), \ldots, g_n(x)]^T\) and \(g_j(x_j(t)) = f_j(x_j(t) + y_j^*) - f_j(y_j^*)\) with \(g_j(0) = 0\), \(\forall j\). As in many applications, one can assumed that each activation function \(g(x)\) satisfies the following sector condition:

\[
\begin{align*}
g_j(x_j(t))(g_j(x_j(t)) - k_j x_j(t)) &\leq 0, \quad j = 1, 2, \ldots, n,
\end{align*}
\]

where \(k_j \in \mathbb{R}^+\).

The following fact and lemmas will be used for deriving main result.

Fact 1 (Schur complement). Given constant symmetric matrices \(\Sigma_1, \Sigma_2, \Sigma_3\) where \(\Sigma_1 = \Sigma_1^T\) and \(0 < \Sigma_2 = \Sigma_2^T\), then \(\Sigma_1 + \Sigma_2^T \Sigma_2^{-1} \Sigma_3 < 0\) if and only if

\[
\begin{bmatrix}
\Sigma_1 & \Sigma_3^T \\
\Sigma_3 & -\Sigma_2
\end{bmatrix} < 0, \quad \text{or} \quad
\begin{bmatrix}
-\Sigma_2 & \Sigma_1 \\
\Sigma_3^T & \Sigma_3
\end{bmatrix} < 0.
\]

Lemma 1 [17]. Consider an operator \(\mathcal{D}(\cdot) : \mathcal{C}_{n,h} \to \mathbb{R}^n\) with \(\mathcal{D}(x_t) = x(t) + \hat{B} \int_{t-h}^t x(s)ds\), where \(x(t) \in \mathbb{R}^n\) and \(\hat{B} \in \mathbb{R}^{n \times n}\). For a given scalar \(\delta\), where \(0 < \delta < 1\), if a positive definite symmetric matrix \(M\) exists, such that

\[
\begin{bmatrix}
-\delta M & h\hat{B}^T M \\
hM \hat{B} & -M
\end{bmatrix} < 0
\]

holds, then the operator \(\mathcal{D}(x_t)\) is stable.

Lemma 2 [16]. For any constant matrix \(\Sigma \in \mathbb{R}^{n \times n}\), \(\Sigma = \Sigma^T > 0\), scalar \(\gamma > 0\), vector function \(\omega : [0, \gamma] \to \mathbb{R}^n\) such that the integrations concerned are well defined, then

\[
\left(\int_0^\gamma \omega(s)ds\right)^T \Sigma \left(\int_0^\gamma \omega(s)ds\right) \leq \gamma \int_0^\gamma \omega^T(s)\Sigma \omega(s)ds.
\]
3. Main result

Here, a stability condition for global asymptotic stability of system (2) will be presented. First, define a new mathematical operator \( \mathcal{D}(x_i) : \mathcal{C}_{x_i} \rightarrow \mathbb{R}^n \) as

\[
\mathcal{D}(x_i) = x(t) + \int_{t-h}^{t} Gx(s)ds,
\]

where \( x_t = x(t + s), s \in [-h, 0] \) and \( G \) is a free parameter to be determined later.

From the definition of \( \mathcal{D}(x_i) \), the transformed system is

\[
\dot{\mathcal{D}}(x_i) = - (A - G)x(t) + Wg(x(t)) + W_1g(x(t - h)) - Gx(t - h),
\]

Then we have the following theorem.

**Theorem 1.** For given \( K = \text{diag}\{k_1, k_2, \ldots, k_n\} \), \( \gamma \) and \( h \), the equilibrium point of Eq. (2) is globally asymptotically stable if there exist positive definite matrices \( P, R, N, Q, D = \text{diag}\{d_1, \ldots, d_n\} \), and any matrices \( Y, S_i, i = 1, \ldots, 6 \), satisfying the following LMIs:

\[
\begin{bmatrix}
\Pi_1 & hY^T & -Y + S_2 - S_1^T & \Pi_3 & \Pi_5 & \Pi_8 & S_6 - S_1^T \\
\star & -P & 0 & 0 & 0 & 0 & 0 \\
\star & \star & -R - S_2 - S_2^T & -S_3 & -S_4 & -Y^T - S_5 & -S_6 - S_2^T \\
\star & \star & \star & \Pi_2 & \Pi_6 & W^TP & -S_4^T \\
\star & \star & \star & \star & \star & \star & -P & -S_3^T \\
\star & \star & \star & \star & \star & \star & \star & \Pi_7 \\
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
-P & hY^T \\
\star & -P \\
\end{bmatrix} < 0,
\]

where

\[
\begin{align*}
\Pi_1 &= -PA - A^TP + Y + Y^T + R + hA^TN + S_1 + S_1^T, \\
\Pi_2 &= -2DK^{-1}A + Q + DW + W^TD + hW^TNW, \\
\Pi_3 &= PW + S_3 - hA^TNW, \\
\Pi_4 &= -Q + hW_1^TNW, \\
\Pi_5 &= PW_1 + S_4 - hA^TNW_1, \\
\Pi_6 &= DW_1 + hW_1^TNW_1, \\
\Pi_7 &= -(1/h)N - S_6 - S_6^T, \\
\Pi_8 &= -A^TP + Y^T + S_5.
\end{align*}
\]

**Proof.** For \( P > 0, Q > 0, R > 0, N > 0 \) and a positive scalar \( d_i \), consider the following Lyapunov functional

\[
V = \mathcal{D}^T(x_i)P \mathcal{D}(x_i) + 2 \sum_{i=1}^{n} d_i \int_{0}^{x_i(t)} g_i(s)ds + \int_{t-h}^{t} g^T(x(s))Qg(x(s))ds \\
+ h \int_{t-h}^{t} \int_{s}^{t} x^T(s)GPGx(s)dsds + \int_{t-h}^{t} x^T(s)Rx(s)ds + \int_{t-h}^{t} (x - t + h)x^T(s)N\dot{x}(s)ds.
\]
The time derivative of $V$ along the trajectory of systems (3) and (8) are as follows:

$$
\dot{V} = 2\left( x(t) + \int_{t-h}^{t} Gx(s)ds \right)^{T} P\left( -(A - G)x(t) + Wg(x(t)) + W_{1}g(x(t - h)) - Gx(t - h) \right) \\
+ 2g^{T}(x(t))D[-Ax(t) + Wg(x(t)) + W_{1}g(x(t - h))] + g^{T}(x(t))Qg(x(t)) - g^{T}(x(t - h))Qg(x(t - h)) \\
+ h^{2}x^{T}(t)G^{T}PGx(t) - h \int_{t-h}^{t} x^{T}(s)G^{T}PGx(s)ds + x^{T}(t)Rx(t) - x^{T}(t - h)Rx(t - h) \\
+ h\dot{x}^{T}(t)N\dot{x}(t) - \int_{t-h}^{t} \dot{x}^{T}(s)N\dot{x}(s)ds.
$$

(12)

Here note that

$$
-2g^{T}(x(t))DK^{-1}Ag(x(t)), \\
- h \int_{t-h}^{t} x^{T}(s)G^{T}PGx(s)ds \leq \left( \int_{t-h}^{t} Gx(s)ds \right)^{T} P \left( \int_{t-h}^{t} Gx(s)ds \right), \\
- h \int_{t-h}^{t} \dot{x}^{T}(s)N\dot{x}(s)ds \leq \left( \int_{t-h}^{t} \dot{x}(s)ds \right)^{T} N \left( \int_{t-h}^{t} \dot{x}(s)ds \right).
$$

(13), (14), (15)

where Lemma 2 is used in second and third inequalities.

For any appropriate dimensional matrices $S_{i}$, $i = 1, 2, \ldots, 6$, the following equation holds:

$$
2 \left[ x(t) - x(t - h) - \int_{t-h}^{t} \dot{x}(s)ds \right]^{T} \left[ S_{1}x + S_{2}x(t - h) + S_{3}g(x(t)) \\
+ S_{4}g(x(t - h)) + S_{5} \int_{t-h}^{t} Gx(s)ds + S_{6} \int_{t-h}^{t} \dot{x}(s)ds \right] = 0.
$$

(16)

Utilizing Eqs. (13)–(16), then we obtain the following inequality:

$$
\dot{V} \leq z^{T}(t)
$$

$$
\begin{bmatrix}
(1, 1) & -PG + S_{2} - S_{1}^{T} & \Pi_{3} & \Pi_{5} & -(A - G)^{T}P + S_{5} & S_{6} - S_{1}^{T} \\
\ast & -R - S_{2} - S_{1}^{T} & -S_{3} & -S_{4} & -G^{T}P - S_{5} & -S_{6} - S_{2}^{T} \\
\ast & \ast & \Pi_{2} & \Pi_{6} & W^{T}P & -S_{3}^{T} \\
\ast & \ast & \ast & \Pi_{4} & W_{1}^{T}P & -S_{4}^{T} \\
\ast & \ast & \ast & \ast & -P & -S_{5}^{T} \\
\ast & \ast & \ast & \ast & \ast & \Pi_{7}
\end{bmatrix}
$$

\[ z(t), \]  

(17)

where

$$
z(t) = \left[ x^{T}(t) \quad x^{T}(t - h) \quad g^{T}(x(t)) \quad g^{T}(x(t - h)) \quad \left( \int_{t-h}^{t} Gx(s)ds \right)^{T} \quad \left( \int_{t-h}^{t} \dot{x}(s)ds \right)^{T} \right]^{T}, \\
(1, 1) = -P(A - G) - (A - G)^{T}P + h^{2}G^{T}PG + R + hA^{T}NA + S_{1} + S_{1}^{T}.
$$

Here, by defining $Y$ as $Y = PG$, we have

$$
\dot{V} \leq z^{T}(t)
$$

$$
\begin{bmatrix}
\Pi_{1} + h^{2}Y^{T}P^{-1}Y & -Y + S_{2} - S_{1}^{T} & \Pi_{3} & \Pi_{5} & \Pi_{8} & S_{6} - S_{1}^{T} \\
\ast & -R - S_{2} - S_{1}^{T} & -S_{3} & -S_{4} & -Y^{T}S_{5} & -S_{6} - S_{2}^{T} \\
\ast & \ast & \Pi_{2} & \Pi_{6} & W^{T}P & -S_{3}^{T} \\
\ast & \ast & \ast & \Pi_{4} & W_{1}^{T}P & -S_{4}^{T} \\
\ast & \ast & \ast & \ast & -P & -S_{5}^{T} \\
\ast & \ast & \ast & \ast & \ast & \Pi_{7}
\end{bmatrix}
$$

\[ z(t) \equiv z^{T}(t)Mz(t). \]  

(18)
If $\mathcal{M}$ is a negative definite matrix, then $\dot{V} < 0$. By Fact 1 (Schur complement), the inequality $\mathcal{M} < 0$ is equivalent to the LMI (9). Also, the inequality (10) is equivalent to

$$
\begin{bmatrix}
-P & hG^TP \\
\star & -P
\end{bmatrix} < 0.
$$

If the above inequality holds, then we can prove that a positive scalar $\delta$ which is less than one exists such that

$$
\begin{bmatrix}
-\delta P & hG^TP \\
\star & -P
\end{bmatrix} < 0
$$

according to matrix theory. Therefore, from Lemma 1, if the inequality (10) holds, then operator $\mathcal{D}(x_i)$ is stable. According to Theorem 9.8.1 in [18], we conclude that if matrix inequalities (9) and (10) hold, then the equilibrium point of system (2) is asymptotically stable. This completes our proof.

**Remark 1.** The criterion given in Theorem 1 is delay-dependent. It is well known that the delay-dependent criteria are less conservative than delay-independent criteria when the delay is small. The solutions of Theorem 1 can be obtained by solving the eigenvalue problem with respect to solution variables, $P$, $Q$, $N$, $R$, $S$, $Y$, which is a convex optimization problem [14]. In this paper, we utilize Matlab's LMI Control Toolbox [15] which implements interior-point algorithm. This algorithm is significantly faster than classical convex optimization algorithms [14].

**Remark 2.** By iteratively solving the LMIs given in Theorem 1 with respect to $h$, one can find the maximum upper bound of time delay $h$ for guaranteeing asymptotic stability of system (2).

**Remark 3.** Recently, by many researchers, some remarkable stability criteria for DCNN with $A = I$ have been presented as

- Arik and Tavsanoglu [8]:
  \begin{align}
  \begin{cases}
  -(W + W^T) > 0, \\
  \|W_1\| \leq 1,
  \end{cases}
  \end{align}

- Liao and Wang [9]:
  \begin{align}
  \begin{cases}
  -(W + W^T + \beta I) > 0 \\
  \|W_1\| \leq \sqrt{\bar{\beta} + \bar{\beta}}
  \end{cases}
  \end{align}

- Cao [10]:
  \begin{align}
  \begin{cases}
  -(W + W^T + \beta I) > 0, \\
  \|W_1\| \leq \sqrt{\bar{\beta} + \bar{\beta}}
  \end{cases}
  \end{align}

- Arik [11]:
  \begin{align}
  \begin{cases}
  -(W + W^T + \beta I) > 0 \\
  \|W_1\| \leq \sqrt{2\beta}
  \end{cases}
  \end{align}

- Arik [12]:
  \begin{align}
  \begin{cases}
  -(D\bar{W} + W^TD + K) > 0, \\
  -2D - K + I + DWW_1^TD \leq 0
  \end{cases}
  \end{align}

- Singh [13]:
  \begin{align}
  -D + DW + W^TD + W_1^TDW_1 < 0,
  \end{align}

where $\alpha$ is the minimum eigenvalue of $-(W + W^T + \beta I)$, $D = [d_i]$ is a positive definite matrix, and $\bar{K} = [\bar{k}_{ij}]$ is a symmetric positive definite matrix.

In the following, we will compare our result with above ones to show the effectiveness of ours.

**Example 1.** Consider a DCNN with

$$A = I, \quad W = \begin{bmatrix} 0 & 0.9 \\ -1 & -1 \end{bmatrix}, \quad W_1 = 1.1 \times \begin{bmatrix} 0.6 & -1 \\ 1 & 0 \end{bmatrix},$$

and the activation function $g_i(x) = 0.5(|x + 1| - |x - 1|)$. 


It is easy to verify that each of the criteria given (21)–(26! fails. However, by applying Theorem 1, one can see that our criterion given in Theorem 1 is feasible for all *h* > 0. It means that for this example our criteria show that the system is actually delay-independent stable, which is not possible to know from the existing delay-dependent criteria (21)–(26). For reference, when *h* = 10, the LMI solutions of Theorem for this example are as follows:

\[
P = \begin{bmatrix} 0.6242 & 0.1217 \\ 0.1217 & 1.0806 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.0027 & 0.0008 \\ 0.0008 & 0.0009 \end{bmatrix},
\]

\[
Q = \begin{bmatrix} 4.2414 & -2.3966 \\ -2.3966 & 7.7932 \end{bmatrix}, \quad D = \begin{bmatrix} 4.4534 & 0 \\ 0 & 3.5046 \end{bmatrix},
\]

\[
R = \begin{bmatrix} 0.4675 & 0.0674 \\ 0.0674 & 0.7998 \end{bmatrix}, \quad N = \begin{bmatrix} 0.0033 & 0.0013 \\ 0.0013 & 0.0060 \end{bmatrix},
\]

\[
S_1 = \begin{bmatrix} -0.2196 & 0.0410 \\ 0.0410 & -0.0565 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.3145 & -0.0098 \\ -0.0098 & 0.2834 \end{bmatrix},
\]

\[
S_3 = \begin{bmatrix} 0.0363 & -0.1406 \\ -0.1406 & 0.3080 \end{bmatrix}, \quad S_4 = \begin{bmatrix} -0.1701 & 0.2165 \\ -0.2165 & 0.3985 \end{bmatrix},
\]

\[
S_5 = \begin{bmatrix} 0.2097 & 0.0401 \\ 0.0400 & 0.3597 \end{bmatrix}, \quad S_6 = \begin{bmatrix} 0.4710 & 0.0128 \\ 0.0125 & 0.5196 \end{bmatrix}.
\]

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**References**


