New delay-dependent robust stability criterion for uncertain neural networks with time-varying delays

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A R T I C L E   I N F O

Keywords:
Asymptotic stability
Delay
Linear matrix inequality (LMI)
Lyapunov method
Neural networks (NNs)

A B S T R A C T

In this paper, the problem of delay-dependent stability of uncertain neural networks with time-varying delays is considered. The parameter uncertainties are time-varying and assumed to be bounded in magnitude. By constructing a new augmented Lyapunov functional, a delay-dependent stability criterion for the network is derived in terms of LMI (linear matrix inequality) which can be easily solved by various convex optimization algorithms. Three numerical examples are included to show the effectiveness of proposed criterion.

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1. Introduction

The stability analysis of neural networks (NNs) has been extensively investigated since NNs have been widely applied to various fields such as pattern recognition, associative memories, signal processing, fixed-point computations, and so on. The applications of NNs heavily depend on the dynamic behavior of the equilibrium point. For related works, see [1–12] and reference therein. In the processing of storage and transmission of information, time delays often occur due to the finite switching speed of amplifiers in electronic networks or finite speed for signal propagation in biological networks [2–4]. The time delays may cause divergence, oscillation, and even instability. Therefore, considerable efforts have been done to stability analysis of delayed neural networks (DNNs) in recent years [5–12].

In deriving the delayed-dependent stability criterion of dynamic system with time delays, which is less conservative than delay-independent one when the value of the size of delays are small, the main concern is to enlarge the feasibility region of stability criteria or to get the maximum allowable bound of time delays for guaranteeing the stability. To do this, a new bounding technique was proposed in Park [13] to reduce the conservatism of the stability criteria by introducing variables in cross-terms. Model transformations such as neutral model one [14] and parameterized neutral one [7] were utilized to enlarge the feasibility region of the proposed stability criteria. Recently, the method of utilizing zero equations including free weighting matrices were suggested to increase the stability region [10–12,15]. Augmented Lyapunov functional were proposed to derive improved stability criteria [11,16].

In this paper, we revisit the problem of delay-dependent stability analysis of DNNs with time-varying delays and uncertainties. The uncertainties are assumed to be bounded in magnitude. By constructing a new augmented Lyapunov functional and utilizing zero equations including free-weighting matrices, a new delay-dependent criterion is derived in terms of LMI which can be solved efficiently by using the interior-point algorithms [17]. Three numerical examples are shown to support that our results are less conservative than those of the existing literature.

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Notation: $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrix. $\| \cdot \|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices $X$ and $Y$, the notation $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite, (respectively, nonnegative). $\text{diag}\{\cdots\}$ denotes the block diagonal matrix. ★ represents the elements below the main diagonal of a symmetric matrix.

2. Problem statements

Consider the following uncertain NNs with discrete time-varying delays:

$$y(t) = -(A + \Delta A(t))y(t) + (W_0 + \Delta W_0(t))g(y(t)) + (W_1 + \Delta W_1(t))g(y(t - h(t))) + b,$$  

(1)

where $y(t) = [y_1(t), \ldots, y_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector, $n$ denotes the number of neurons in a neural network, $g(y(t)) = [g_1(y_1(t)), \ldots, g_n(y_n(t))]^T \in \mathbb{R}^n$ denotes the neuron activation function, $g(y(t - h(t))) = [g_1(y_1(t - h(t))), \ldots, g_n(y_n(t - h(t)))]^T$ is a positive diagonal matrix, $W_0 = (w_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$, and $W_1 = (w_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$ are the interconnection matrices representing the weight coefficients of the neurons, $b = [b_1, b_2, \ldots, b_n]^T \in \mathbb{R}^n$ means a constant input vector, and $\Delta A(t), \Delta W_0(t), \Delta W_1(t)$, are the uncertainties of system matrices of the form:

$$[\Delta A(t) \; \Delta W_0(t) \; \Delta W_1(t)] = DF(t)[E_0 \; E_1 \; E_2],$$

(2)

where the time-varying nonlinear function $F(t)$ satisfies:

$$F^T(t)F(t) \leq I \quad \forall t \geq 0.$$  

(3)

The delays, $h(t)$, are time-varying continuous functions that satisfies:

$$0 \leq h(t) \leq h_u, \quad h(t) \leq h_0.$$  

(4)

where $h_u$ is a positive scalar and $h_0$ is any constant one.

The activation functions, $g_i(y_i(t))$, $i = 1, \ldots, n$, are assumed to be nondecreasing, bounded and globally Lipschitz; that is,

$$0 \leq g_i'(\xi_i) - g_i'(\zeta_i) \leq l_i, \quad \xi_i, \; \zeta_i \in \mathbb{R}, \; \xi_i \neq \zeta_i, \; i = 1, \ldots, n,$$

(5)

where $l_i$, $i = 1, \ldots, n$ are positive constants.

Note that by using the Brouwer’s fixed-point theorem [5], it can be easily proven that there exists at least one equilibrium point for Eq. (1).

For simplicity, in stability analysis of the system (1), the equilibrium point $y^* = [y_1^*, \ldots, y_n^*]^T$ is shifted to the origin by utilizing the transformation $x(\cdot) = y(\cdot) - y^*$, which leads the system (1) to the following form:

$$\dot{x}(t) = -(A + \Delta A(t))x(t) + (W_0 + \Delta W_0(t))\dot{g}(x(t)) + (W_1 + \Delta W_1(t))\dot{g}(x(t - h(t))),$$

(6)

where $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the transformed system.

Note that $\dot{g}(x(t)) = [\dot{g}_1(x_1(t)), \ldots, \dot{g}_n(x_n(t))]^T$ and $\dot{g}_i(x_i(t)) = g_i(x_i(t) + y_i^*) - g_i(y_i^*)$ with $f_j(0) = 0$ ($j = 1, \ldots, n$).

From (5), note that $\dot{g}_i(\cdot)$ satisfies the following condition:

$$0 \leq \dot{g}_i(\xi_i) - \dot{g}_i(\zeta_i) \leq l_i \quad \forall \xi_i \neq \zeta_i, \; j = 1, \ldots, n,$$

(7)

which is equivalent to:

$$\dot{f}_j(\xi_j) - \dot{f}_j(\zeta_j) \leq 0, \quad \dot{f}_j(0) = 0, \; j = 1, \ldots, n.$$  

(8)

System (6) can be written as

$$\dot{x}(t) = -Ax(t) + W_0f(x(t)) + W_1f(x(t - h(t))) + Dp(t),$$

$$p(t) = F(t)q(t),$$

$$q(t) = -E_0x(t) + E_1f(x(t)) + E_2f(x(t - h(t))),$$

The objective of this paper is to investigate the delay dependent stability analysis of system (9) which will be conducted in Section 3.

Before deriving our main results, we state the following fact, and lemma.

**Fact 1** (Schur complement). Given constant symmetric matrices $\Sigma_1, \Sigma_2, \Sigma_3$ where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if:

$$\begin{bmatrix}
\Sigma_1 & \Sigma_3^T \\
\Sigma_3 & -\Sigma_2
\end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix}
-\Sigma_2 & \Sigma_3 \\
\Sigma_3^T & \Sigma_1
\end{bmatrix} < 0.$$

**Lemma 1** [18]. For any constant matrix $M \in \mathbb{R}^{m \times n}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $x : [0, \gamma] \rightarrow \mathbb{R}^n$ such that the integrations concerned are well defined, then
\[
\left( \int_0^\gamma x(s)ds \right)^T M \left( \int_0^\gamma x(s)ds \right) \leqslant \gamma \int_0^\gamma x^T(s)Mx(s)ds.
\]

3. Main results

In this section, we propose a new stability criterion for NNs with time-varying delays described by Eq. (6). Before stating our main results, the notation of several matrices is defined for simplicity in Appendix A. Now, we have the following theorem.

**Theorem 1.** For given \( h_U, h_D, \) and \( L \), the equilibrium point of (1) is globally asymptotically stable if there exist positive diagonal matrices \( K = \text{diag}(k_1, k_2, \ldots, k_n) \), \( H_i = \text{diag}(h_{1i}, h_{2i}, \ldots, h_{ni}) \) \((i = 1, \ldots, 3)\), positive definite matrices \( P_i \), \( R_i \) \((i = 1, 2)\), \( H \), \( Q_i \) \((i = 1, 2)\), \( M_{ij} \) \((i = 1, 2, 3)\), \( N_{ij} \) \((i = 1, 2)\), \( G_{ii} \) \((i = 1, 2, 3)\), \( X_i \) \((i = 2, 4, 5, 7, 8, 9)\) satisfying the following LMIs:

\[
\begin{bmatrix}
\Sigma & \Psi^T H \\
\star & -H
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
X & P^T U_1 \\
\star & Q_1
\end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
X & P^T U_2 \\
\star & Q_3
\end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
S_1 & S_2 & S_4 \\
\star & S_3 & S_5 \\
\star & \star & S_6
\end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
M_{11} & M_{12} & M_{12} \\
\star & M_{22} & M_{23} \\
\star & \star & M_{33}
\end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
N_{11} & N_{12} \\
\star & N_{22}
\end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
G_{11} & G_{12} \\
\star & G_{22}
\end{bmatrix} > 0.
\]

where \( \Sigma \) is defined in Appendix A.

\[
\Psi = [-E_0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ E_1 \ E_2 \ 0 \ 0 \ 0 \ 0],
\]

\[
U_1 = \begin{bmatrix}
0 \\
0 \\
I \\
0
\end{bmatrix}, \quad U_2 = \begin{bmatrix}
0 \\
0 \\
0 \\
I
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
P_1 & 0 & 0 & 0 \\
P_2 & P_3 & P_4 & P_5 \\
P_6 & P_7 & 0 & P_8 \\
P_9 & P_{10} & 0 & P_{11}
\end{bmatrix},
\]

and

\[
X = \begin{bmatrix}
X_1 & X_2 & X_4 & X_7 \\
\star & X_3 & X_5 & X_8 \\
\star & \star & X_6 & X_9 \\
\star & \star & \star & X_{10}
\end{bmatrix}.
\]

**Proof.** Firstly, let us define the following two zero equations:

\[
\begin{align*}
z_1(t) & : \quad x(t) - x(t - h(t)) - \int_{t-h(t)}^{t} x(s)ds = 0, \\
z_2(t) & : \quad x(t - h(t)) - x(t - h_U) - \int_{t-h_U}^{t-h(t)} x(s)ds = 0.
\end{align*}
\]
For positive definite matrices $P_i$, $K$, $R_i$ ($i = 1, 2$), $Q_i$ ($i = 1, 2$), $M_i$ ($i = 1, 2, 3$), $N_i$ ($i = 1, 2$), $G_i$ ($i = 1, 2$), and any matrices $P_i$ ($i = 2, 3, \ldots, 11$), $M_{12}$, $M_{13}$, $M_{23}$, $N_{12}$, $G_{12}$, let us consider the Lyapunov–Krasovskii functional candidate:

$$V = \sum_{i=1}^{n} V_i(t),$$

where

$$V_i(t) = \zeta_i(t)E\zeta_i(t),$$

$$V_2(t) = 2 \sum_{i=1}^{n} k_i \int_{0}^{t} f_i(s)ds + \int_{t-h(t)}^{t} x^T(s)R_1x(s)ds + \int_{t-h_u}^{t} x^T(s)R_2x(s)ds,$$

$$V_3(t) = \int_{t-h_u}^{t} \int_{s}^{t} \left[ \dot{x}^T(u)Q_1\dot{x}(u) + h_0f^T(x(u))Q_2f(x(u)) \right] |dud$,\n
$$V_4(t) = \left[ \int_{t-h_t}^{t} x(t) \right] ^T S_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} S_2 \begin{bmatrix} \int_{t-h_t}^{t} x(t) \dot{x}(t) \end{bmatrix} S_3 \begin{bmatrix} \int_{t-h_t}^{t} x(t) \dot{x}(t) \end{bmatrix},$$

$$V_5(t) = h_0 \int_{t-h_t}^{t} \int_{s}^{t} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} ^T M_{11} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} ds + h_0 \int_{t-h_t}^{t} \int_{s}^{t} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} ^T G_{11} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} ds.$$

Here, $\zeta_i(t)$, and $E$ in $V_1$ are defined as

$$\zeta_i(t) = [x^T(t)x(t)]r(t)b^T(t),$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By differentiating $V_1$, we obtain the time-derivative of $V_1$ as

$$V_1 = 2\dot{\zeta}_1(t)^T P \dot{\zeta}_1(t) = 2\dot{\zeta}_1(t)^T P \begin{bmatrix} \dot{x}(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \dot{\zeta}_1(t)^T (P^T \Gamma + \Gamma^T P) \dot{\zeta}_1(t) + 2\dot{\zeta}_1(t)^T P U_1 W_0f(x(t)) + 2\dot{\zeta}_1(t)^T P U_2 W_0f(x(t-h(t)))$$

$$- 2\dot{\zeta}_1(t)^T P U_1 \int_{t-h(t)}^{t} \dot{x}(s)ds - 2\dot{\zeta}_1(t)^T P U_2 x(t-h(t))$$

where

$$\Gamma = \begin{bmatrix} 0 & 0 & I & 0 \\ -A & 0 & -I & D \\ I & -I & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

and

$$U_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
By using Lemma 1, we can obtain the upper bound of time-derivative $V_3$ as follows:
\[
V_3 \leq h_0 x^T(t)Q_1x(t) - \int_{t-h(t)}^{t} x^T(s)Q_1x(s)ds - \int_{t-h(t)}^{t} x^T(s)Q_2x(s)ds + h_0^2 f^T(x(t))Q_2f(x(t))
- \left( \int_{t-h(t)}^{t} f(x(s))ds \right)^T Q_2 \left( \int_{t-h(t)}^{t} f(x(s))ds \right) - \left( \int_{t-h(t)}^{t} f(x(s))ds \right)^T Q_2 \left( \int_{t-h(t)}^{t} f(x(s))ds \right).
\]
(29)

By calculating $V_4$, we have:
\[
V_4 = 2 \begin{bmatrix}
\int_{t-h(t)}^{t} x(s)ds + \int_{t-h(t)}^{t} x(s)ds \\
\int_{t-h(t)}^{t} f(x(s))ds + \int_{t-h(t)}^{t} f(x(s))ds
\end{bmatrix}
\begin{bmatrix}
S_1 & S_2 & S_3 \\
S_4 & S_5 & S_6
\end{bmatrix}
\begin{bmatrix}
\dot{x}(t) \\
x(t) - x(t - h(t))
\end{bmatrix}
= \zeta^T(t) \Xi_1 \zeta(t),
\]
(30)

where
\[
\zeta^T(t) = \left[ x^T(t)x^T(t - h(t))x^T(t - h(t)) \right] \begin{bmatrix}
\int_{t-h(t)}^{t} x(s)ds \int_{t-h(t)}^{t} x(s)ds \int_{t-h(t)}^{t} f(x(s))ds \int_{t-h(t)}^{t} f(x(s))ds
\end{bmatrix}
\begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
\dot{x}(t) \\
x(t) - x(t - h(t))
\end{bmatrix}
\]

and
\[
\Xi_1 = \begin{bmatrix}
S_2 + S_2 & 0 & -S_1 & S_3 & S_3 & S_4 & 0 & -S_4 & S_5 & S_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -S_1 & -S_1 & S_3 & S_3 & S_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Calculating $V_5$ leads to
\[
V_5 = h_0^2 \begin{bmatrix}
x(t) \\
f(x(t))
\end{bmatrix}^T \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
x(t) \\
f(x(t))
\end{bmatrix} - h_0 \int_{t-h(t)}^{t} \begin{bmatrix}
x(s) \\
f(x(s))
\end{bmatrix}^T \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
x(s) \\
f(x(s))
\end{bmatrix} ds - h_U
\times \int_{t-h(t)}^{t} \begin{bmatrix}
x(s) \\
x(s)
\end{bmatrix}^T \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
x(s) \\
x(s)
\end{bmatrix} ds
\leq h_0^2 \begin{bmatrix}
x(t) \\
f(x(t))
\end{bmatrix}^T \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
x(t) \\
f(x(t))
\end{bmatrix} - \begin{bmatrix}
\int_{t-h(t)}^{t} x(s)ds \\
\int_{t-h(t)}^{t} f(x(s))ds
\end{bmatrix}^T \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
\int_{t-h(t)}^{t} x(s)ds \\
\int_{t-h(t)}^{t} f(x(s))ds
\end{bmatrix}
- \begin{bmatrix}
\int_{t-h(t)}^{t} x(s)ds \\
\int_{t-h(t)}^{t} f(x(s))ds
\end{bmatrix}^T \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
\int_{t-h(t)}^{t} x(s)ds \\
\int_{t-h(t)}^{t} f(x(s))ds
\end{bmatrix}
= \zeta^T(t) \Xi_2 \zeta(t),
\]
(33)
where

\[
\Xi_2 = \begin{bmatrix}
  h_{ij}^2 M_{11} - M_{22} & M_{22} & 0 & h_{ij}^2 M_{12} & -M_{12}^T & 0 & h_{ij}^2 M_{13} & 0 & 0 & -M_{23} & 0 & 0 \\
  \star & -2M_{22} & M_{22} & 0 & M_{12}^T & -M_{12}^T & 0 & 0 & 0 & M_{23} & -M_{23} & 0 \\
  \star & \star & -M_{22} & 0 & 0 & M_{12}^T & 0 & 0 & 0 & 0 & M_{23} & 0 \\
  \star & \star & \star & h_{ij}^2 M_{22} & 0 & 0 & h_{ij}^2 M_{23} & 0 & 0 & 0 & 0 & 0 \\
  \star & \star & \star & \star & -M_{11} & 0 & 0 & 0 & 0 & -M_{13} & 0 & 0 \\
  \star & \star & \star & \star & \star & \star & \star & h_{ij}^2 M_{33} & 0 & 0 & 0 & 0 \\
  \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -M_{33} & 0 \\
  \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & 0
\end{bmatrix}
\]  

(34)

and Lemma 1 was utilized in obtaining an upper bound of \( \bar{V}_5 \).

Finally, an upper bound of \( \bar{V}_6 \) can be obtained as

\[
\bar{V}_6 \leq \begin{bmatrix}
  x(t) \\
  f(x(t))
\end{bmatrix}^T \begin{bmatrix}
  N_{11} & N_{12} \\
  \star & N_{22}
\end{bmatrix} \begin{bmatrix}
  x(t) \\
  f(x(t))
\end{bmatrix} - (1 - h_0) \begin{bmatrix}
  x(t - h(t)) \\
  f(x(t - h(t)))
\end{bmatrix}^T \begin{bmatrix}
  \star & N_{22}
\end{bmatrix} \begin{bmatrix}
  x(t - h(t)) \\
  f(x(t - h(t)))
\end{bmatrix}
\]

\[+
\begin{bmatrix}
  x(t) \\
  f(x(t))
\end{bmatrix}^T \begin{bmatrix}
  G_{11} & G_{12} \\
  \star & G_{22}
\end{bmatrix} \begin{bmatrix}
  x(t - h_0) \\
  f(x(t - h_0))
\end{bmatrix} - \begin{bmatrix}
  x(t - h_0) \\
  f(x(t - h_0))
\end{bmatrix}^T \begin{bmatrix}
  \star & G_{22}
\end{bmatrix} \begin{bmatrix}
  x(t - h_0) \\
  f(x(t - h_0))
\end{bmatrix} = \zeta^T(t) \Xi_3 \zeta(t).
\]

(35)

where

\[
\Xi_3 = \begin{bmatrix}
  N_{11} + G_{11} & 0 & 0 & 0 & 0 & N_{12} + G_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
  \star & -(1 - h_0)N_{11} & 0 & 0 & 0 & 0 & -(1 - h_0)N_{12} & 0 & 0 & 0 \\
  \star & \star & -G_{11} & 0 & 0 & 0 & 0 & -G_{12} & 0 & 0 \\
  \star & \star & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \star & \star & \star & \star & 0 & 0 & 0 & 0 & 0 & 0 \\
  \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & 0 \\
  \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & 0
\end{bmatrix}
\]  

(36)

Here note that Eq. (7) means that:

\[
f_j(x_j(t)) \big| f_j(x_j(t)) - l_j x_j(t) \big| \leq 0 \quad (j = 1, \ldots, n),
\]

(37)

\[
f_j(x_j(t - h(t))) \big| f_j(x_j(t - h(t))) - l_j x_j(t - h(t)) \big| \leq 0 \quad (j = 1, \ldots, n),
\]

(38)

and

\[
f_j(x_j(t - h_0)) \big| f_j(x_j(t - h_0)) - l_j x_j(t - h_0) \big| \leq 0 \quad (j = 1, \ldots, n).
\]

(39)

From three inequalities (37)–(39), for any diagonal positive matrices \( H_1 = \text{diag}\{h_{11}, \ldots, h_{nn}\} \), \( H_2 = \text{diag}\{h_{21}, \ldots, h_{2n}\} \) and \( H_3 = \text{diag}\{h_{31}, \ldots, h_{3n}\} \), the following inequalities hold:

\[
0 \leq -2 \sum_{j=1}^{n} h_{ij} f_j(x_j(t)) \big| f_j(x_j(t)) - l_j x_j(t) \big| - 2 \sum_{j=1}^{n} h_{ij} f_j(x_j(t - h(t))) \big| f_j(x_j(t - h(t))) - l_j x_j(t - h(t)) \big|
\]

\[
-2 \sum_{j=1}^{n} h_{ij} f_j(x_j(t - h_0)) \big| f_j(x_j(t - h_0)) - l_j x_j(t - h_0) \big|
\]

\[
= 2x^T(t) LH_1 f(x(t)) - 2f^T(x(t))H_1 f(x(t)) + 2x^T(t - h(t)) LH_2 f(x(t - h(t))) - 2f^T(x(t - h(t)))H_2 f(x(t - h(t)))
\]

\[
+ 2x^T(t - h_0) LH_3 f(x(t - h_0)) - 2f^T(x(t - h_0))H_3 f(x(t - h_0)).
\]

(40)
Since the following inequality holds from (3) and (9),
\[ p^T(t)p(t) \leq q^T(t)q(t) = \zeta(t)^T \Psi^T \Psi \zeta(t), \] (41)
there exist a positive matrix \( H \) satisfying the following inequality:
\[ \zeta(t)^T \Psi^T H \Psi \zeta(t) - p^T(t)Hp(t) \geq 0, \] (42)
where \( \Psi \) is defined in Theorem 1.
To derive a less conservative result, we add the following zero equation with positive definite matrix \( X \) defined in (19):
\[ 0 = \int_{t-h_0}^t \zeta(t)X(t)\zeta(t)\text{d}s - \int_{t-h_0}^t \zeta(t)X(t)\zeta(t)\text{d}s = h_0\zeta_1(t)X(t) - \int_{t-h_0}^t \zeta(t)X(t)\zeta(t)\text{d}s - \int_{t-h_0}^t \zeta(t)X(t)\zeta(t)\text{d}s. \] (43)

From (25)–(43) and by applying S-procedure [17], the time derivative of \( V \) has a new upper bound as
\[ \dot{V} \leq \zeta(t)^T P \dot{X} + \dot{f}(t)q(t) - t^T P \dot{X} \zeta(t) = 2 \zeta(t)^T P \dot{U}_2(x(t - h(t))) + 2 \zeta(t)^T P \dot{U}_3 \mathcal{W}_d(x(t)) + 2 \zeta(t)^T P \dot{U}_4 \mathcal{W}_d(x(t)) \]
\[ - 2 \zeta(t)^T U_2 \mathcal{X}(x(t) - h(t)) + 2 \zeta(t)^T U_3 \mathcal{X}(x(t) - h(t)) \]
\[ - x(t)^T R_1 \mathcal{X}(x(t) - h(t)) + h_0 \zeta(t)^T Q_1 \mathcal{X}(x(t) - h(t)) + \zeta(t)^T \Psi^T H \Psi \zeta(t) - p^T(t) H p(t) + h_0 \zeta(t)^T Q_1 \mathcal{X}(x(t) - h(t)) \]
\[ = \zeta^T(t) \left( \Sigma + \Psi^T H \Psi \right) \zeta(t) \] (44)

Using Fact 1, the inequality \( \Sigma + \Psi^T H \Psi < 0 \), is equivalent to the LMI (11). Therefore, if LMI (11)–(17) hold, then system (6) is asymptotically stable by the Lyapunov stability theory. This completes our proof. \( \square \)

**Remark 1.** In (22), we proposed new Lyapunov functional \( V_4(t) \) and \( V_5(t) \) utilizing the states \( \int_{t-h_0}^t f(x(s))\text{d}s \) and \( \int_{t-h_0}^t \zeta(t)X(t)\zeta(t)\text{d}s \) as augmented ones in deriving the stability condition of system (6). Thus, Theorem 1 may improve the stability condition of system (6) with more augmented state variables and free weighting matrices. Through numerical examples, the improvement of stability condition comparing with other ones will be shown.

**Remark 2.** In many cases, \( h_0 \) is unknown. For this circumstance, if we do not consider:
\[ \int_{t-h_0}^t \zeta(t)X(t)\zeta(t)\text{d}s \]
and
\[ \int_{t-h_0}^t f(x(s))\text{d}s \]
in (22), then, a delay-dependent stability criterion for system (9) with \( 0 \leq h(t) \leq h_0 \) can be easily obtained.

### 4. Numerical example

**Example 1.** Consider the uncertain DNNs (9) in [10] with:
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1.5 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -2 & 0.5 \\ 0.5 & -2 \end{bmatrix},
\]
\[
D = \begin{bmatrix} 0 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0 \end{bmatrix},
\]
\[
L = \text{diag}(0.4, 0.8).
\]
By applying Theorem 1 to the above system via Matlab LMI control Toolbox [19], we obtained the maximum delay bounds for different values of $h_D$ as listed in Table 1. From Table 1, one can see Theorem 1 in this paper provides improved delay bounds than those in [10].

Example 2. Consider the following nominal DNNs [12]:

$$\dot{x}(t) = -Ax(t) + W_0g(x(t)) + W_1g(x(t-h(t))).$$

where

$$A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ -1.0311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}.$$

Table 2 shows the results of delay bounds for different values of $h_D$ by applying Theorem 1 to the above system. From Table 2, it can be seen that Theorem 1 gives larger delay bounds than the recent results in [8,9,11,12].

Example 3. Consider the nominal neural networks with time-varying delays with the parameters:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}.$$

The comparison of our results with the ones in [8,9,12] are shown in Table 3, which shows that Theorem 1 proposed in this paper improves the existing results.

5. Conclusion

In this paper, a novel delay-dependent stability criterion for neural networks with time-varying delays is proposed. To obtain a less conservative result, a new augmented Lyapunov–Krasovskii functional of descriptor form which includes zero equations is proposed in deriving the stability criterion of (1). Through three numerical examples, the effectiveness of the proposed stability criterion is shown.
Appendix A

\[ \Sigma = \Sigma_{ij} \quad (i,j = 1, \ldots, 12), \]

\[ \Sigma_{(1)} = -P_1^2A - A^TP_2 + P_6 + P_9^T + S_2 + S_2^T + h_0Q_{11}^T - M_{22}^T - M_{11}^T - M_{22} + N_{11} + G_{11} + R_1 + R_2 + h_0X_1, \]

\[ \Sigma_{(2)} = -P_6^T + P_9^2 - A^TP_3 + P_7 + M_{22} + h_0X_2, \]

\[ \Sigma_{(3)} = -P_9^T - S_2, \]

\[ \Sigma_{(4)} = P_1 - P_2^2 - A^TP_4 + S_1 + h_0^2Q_{12} + h_0X_4, \]

\[ \Sigma_{(5)} = S_3 - M_{12}^T, \]

\[ \Sigma_{(6)} = S_3, \]

\[ \Sigma_{(7)} = P_1^2W_0 + S_4 + h_0^2Q_{13} + N_{12} + G_{12} + LH_1, \]

\[ \Sigma_{(8)} = P_1^2W_1, \]

\[ \Sigma_{(9)} = -S_4, \]

\[ \Sigma_{(10)} = S_5 - M_{23}, \]

\[ \Sigma_{(11)} = S_5, \]

\[ \Sigma_{(12)} = P_1^2D - A^TP_5 + P_8 + h_0X_7. \]

\[ \Sigma_{(2.1)} = -P_3 - P_1^2 + P_{10} + P_{10}^T - 2M_{22} + (1 - h_0)N_{11} - (1 - h_0)R_1 + h_0X_3, \]

\[ \Sigma_{(2.2)} = -P_{10}^T + M_{22}, \]

\[ \Sigma_{(2.3)} = -P_{10}^T + M_{22}, \]

\[ \Sigma_{(2.4)} = -P_3^T + h_0X_5, \]

\[ \Sigma_{(2.5)} = M_{12}^T, \]

\[ \Sigma_{(2.6)} = -M_{12}^T, \]

\[ \Sigma_{(2.7)} = P_1^2W_0, \]

\[ \Sigma_{(2.8)} = P_1^2W_1 - (1 - h_0)N_{12} + LH_2, \]

\[ \Sigma_{(2.9)} = 0, \]

\[ \Sigma_{(2.10)} = M_{23}, \]

\[ \Sigma_{(2.11)} = -M_{23}, \]

\[ \Sigma_{(2.12)} = P_1^2D - P_8 + P_{11} + h_0X_8, \]

\[ \Sigma_{(3.1)} = -M_{22} - G_{11} - R_2, \]

\[ \Sigma_{(3.2)} = 0, \]

\[ \Sigma_{(3.3)} = -S_3, \]

\[ \Sigma_{(3.4)} = S_3 + M_{12}^T, \]

\[ \Sigma_{(3.5)} = 0, \]

\[ \Sigma_{(3.6)} = 0, \]

\[ \Sigma_{(3.7)} = -G_{12} + LH_3, \]

\[ \Sigma_{(3.8)} = -S_5 + M_{23}, \]

\[ \Sigma_{(3.9)} = -S_5, \]

\[ \Sigma_{(3.10)} = -P_{11}, \]

\[ \Sigma_{(3.11)} = -P_4 - P_4^T + h_0^2Q_{22} + h_0Q_{11} + h_0X_6, \]

\[ \Sigma_{(4.1)} = S_2, \]

\[ \Sigma_{(4.2)} = S_2, \]

\[ \Sigma_{(4.3)} = h_0Q_{23} + K, \]

\[ \Sigma_{(4.4)} = P_1^2W_0 + h_0^2Q_{23} + K, \]

\[ \Sigma_{(4.5)} = P_1^2W_1, \]

\[ \Sigma_{(4.6)} = P_4^2W_0, \]

\[ \Sigma_{(4.7)} = P_4^2W_1, \]

\[ \Sigma_{(4.8)} = 0, \]

\[ \Sigma_{(4.9)} = 0, \]

\[ \Sigma_{(4.10)} = S_4, \]

\[ \Sigma_{(4.11)} = S_4, \]

\[ \Sigma_{(4.12)} = P_4^2D - P_3 + h_0X_9. \]
\[ \Sigma(5.5) = -M_{11}, \]
\[ \Sigma(5.6) = 0, \]
\[ \Sigma(5.7) = S_5, \]
\[ \Sigma(5.8) = 0, \]
\[ \Sigma(5.9) = S_5, \]
\[ \Sigma(5.10) = -M_{13}, \]
\[ \Sigma(5.11) = 0, \]
\[ \Sigma(5.12) = 0, \]
\[ \Sigma(6.6) = -M_{11}, \]
\[ \Sigma(6.7) = S_5, \]
\[ \Sigma(6.8) = 0, \]
\[ \Sigma(6.9) = S_5, \]
\[ \Sigma(6.10) = 0, \]
\[ \Sigma(6.11) = -M_{13}, \]
\[ \Sigma(6.12) = 0, \]
\[ \Sigma(7.7) = h_0^2 M_{33} + N_{22} + G_{22} + h_0^2 Q_2 - 2H_1, \]
\[ \Sigma(7.8) = 0, \]
\[ \Sigma(7.9) = 0, \]
\[ \Sigma(7.10) = S_6, \]
\[ \Sigma(7.11) = S_6, \]
\[ \Sigma(7.12) = W_0^T P_5, \]
\[ \Sigma(8.8) = -(1 - h_0) N_{22} - 2H_2, \]
\[ \Sigma(8.9) = 0, \]
\[ \Sigma(8.10) = 0, \]
\[ \Sigma(8.11) = 0, \]
\[ \Sigma(8.12) = W_1^T P_5, \]
\[ \Sigma(9.9) = -G_{22} - 2H_3, \]
\[ \Sigma(9.10) = -S_6, \]
\[ \Sigma(9.11) = -S_6, \]
\[ \Sigma(9.12) = 0, \]
\[ \Sigma(10.10) = -M_{33} - Q_2, \]
\[ \Sigma(10.11) = 0, \]
\[ \Sigma(10.12) = 0, \]
\[ \Sigma(11.11) = -M_{33} - Q_2, \]
\[ \Sigma(11.12) = 0, \]
\[ \Sigma(12.12) = P_1^T D + D^T P_5 - H + h_0 X_{12}. \]

References