Stability for a competitive Lotka–Volterra system with delays: LMI optimization approach

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Abstract

Using an LMI (linear matrix inequality) optimization approach, a stability criterion is obtained for the local stability of the positive equilibrium of a Lotka–Volterra system with delays.

Keywords: Local stability; Lotka–Volterra system; Delay; LMI

1. Introduction

Stability analysis of population models governed by delay-differential equations has been extensively studied in a number of papers (see the books and papers [1–4] and the references cited therein). In this note, the following Lotka–Volterra type competitive system with discrete delays is considered:

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t)[b_1 - a_{11}x(t - \tau_{11}) - a_{12}y(t - \tau_{12})], \\
\frac{dy(t)}{dt} &= y(t)[b_2 - a_{21}x(t - \tau_{21}) - a_{22}y(t - \tau_{22})],
\end{align*}
\]

where \(x(t)\) and \(y(t)\) stand for densities of the population at time \(t\), respectively, \(b_i, a_{ij}\) are all positive constants, and the delays \(\tau_{ij}\) are positive.

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The initial condition of (1) is given as
\[
\begin{align*}
\begin{cases}
x(s) = \phi_1(s) \geq 0, & -\tau \leq s \leq 0; \ \phi_1(0) > 0, \\
y(s) = \phi_2(s) \geq 0, & -\tau \leq s \leq 0; \ \phi_2(0) > 0, \\
\tau = \max\{\tau_{ij}\}.
\end{cases}
\end{align*}
\]
(2)

Under the condition \(a_{11}/a_{21} > b_1/b_2 > a_{12}/a_{22}\) [5], all positive solutions \(Z(t) = (x(t), y(t))\) of system (1) have unique positive equilibrium \(Z^* = (x^*, y^*)\):
\[
\begin{align*}
x^* &= \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}}, \\
y^* &= \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}.
\end{align*}
\]
(3)

Recently, Zhen and Ma [5] have studied the asymptotic stability of the positive equilibrium of system (1), and sufficient conditions for the stability of the system have been derived using Lyapunov functionals. In this note, our purpose is to seek sharp conditions for the local asymptotic stability of system (1).

Through the article, \(\star\) represents the elements below the main diagonal of a symmetric matrix. The notation \(X > Y\), where \(X\) and \(Y\) are matrices of the same dimensions, means that the matrix \(X - Y\) is positive definite.

2. Main result

Define
\[
\begin{align*}
u(t) &= x(t) - x^*, \\
v(t) &= y(t) - y^*,
\end{align*}
\]
then the variational system of (1) with respect to the positive equilibrium \(Z^*\) is given by the following linearized system [5]:
\[
\begin{align*}
\begin{cases}
\frac{du(t)}{dt} = -a_{11} x^* u(t - \tau_{11}) - a_{12} x^* v(t - \tau_{12}), \\
\frac{dv(t)}{dt} = -a_{21} y^* u(t - \tau_{21}) - a_{22} y^* v(t - \tau_{22}).
\end{cases}
\end{align*}
\]
(4)

Eq. (4) can be rewritten in the following form:
\[
\begin{align*}
\frac{d}{dt} \left[ u(t) - a_{11} \int_{t-\tau_{11}}^t u(s) ds - a_{12} \int_{t-\tau_{12}}^t v(s) ds \right] &= -a_{11} u(t) - a_{12} v(t), \\
\frac{d}{dt} \left[ v(t) - a_{21} \int_{t-\tau_{21}}^t u(s) ds - a_{22} \int_{t-\tau_{22}}^t v(s) ds \right] &= -a_{21} u(t) - a_{22} v(t).
\end{align*}
\]
(5)

Then, we have the following theorem for the local stability of system (1) using the Lyapunov method and the LMI optimization technique.

**Theorem 1.** For given delays \(\tau_{ij}\), if there exist positive scalars \(\varepsilon_1, \varepsilon_2, p_1, p_2, q_1, q_2, q_3, q_4\) satisfying the following LMI:
then the positive equilibrium $Z^*$ of (1) is locally asymptotically stable.

**Proof.** Let us consider the following legitimate Lyapunov functional candidate [1]:

$$V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6$$

where

$$
\begin{align*}
V_1 &= p_1 \left( \frac{u(t)}{x^*} - a_{11} \int_{t-\tau_{11}}^t u(s)ds - a_{12} \int_{t-\tau_{12}}^t v(s)ds \right)^2 \\
V_2 &= p_2 \left( \frac{v(t)}{y^*} - a_{21} \int_{t-\tau_{21}}^t u(s)ds - a_{22} \int_{t-\tau_{22}}^t v(s)ds \right)^2 \\
V_3 &= q_1 \int_{t-\tau_{11}}^t (s - t + \tau_{11}) u^2(s)ds \\
V_4 &= q_2 \int_{t-\tau_{12}}^t (s - t + \tau_{12}) v^2(s)ds \\
V_5 &= q_3 \int_{t-\tau_{21}}^t (s - t + \tau_{21}) u^2(s)ds \\
V_6 &= q_4 \int_{t-\tau_{22}}^t (s - t + \tau_{22}) v^2(s)ds
\end{align*}
$$

with positive scalars $p_1, p_2, q_1, q_2, q_3, q_4$.

Taking the time derivative of $V_i$ along the solution of (5), we have

$$
\begin{align*}
\dot{V}_1 &= 2(-a_{11}u(t) - a_{12}v(t)) p_1 \left( \frac{u(t)}{x^*} - a_{11} \int_{t-\tau_{11}}^t u(s)ds - a_{12} \int_{t-\tau_{12}}^t v(s)ds \right) \\
\dot{V}_2 &= 2(-a_{21}u(t) - a_{22}v(t)) p_2 \left( \frac{v(t)}{y^*} - a_{21} \int_{t-\tau_{21}}^t u(s)ds - a_{22} \int_{t-\tau_{22}}^t v(s)ds \right)
\end{align*}
$$
\[ \dot{V}_3 = \tau_{11} q_1 u^2(t) - q_1 \int_{t-r_{11}}^{t} u^2(s) ds \]
\[ \dot{V}_4 = \tau_{22} q_2 v^2(t) - q_2 \int_{t-r_{22}}^{t} v^2(s) ds \]
\[ \dot{V}_5 = \tau_{12} q_3 v^2(t) - q_3 \int_{t-r_{12}}^{t} v^2(s) ds \]
\[ \dot{V}_6 = \tau_{21} q_4 u^2(t) - q_4 \int_{t-r_{21}}^{t} u^2(s) ds. \] (10)

Here, from the well-known fact
\[ \theta \sigma \int_0^{\sigma} \omega^2(s) ds \geq \theta \left( \int_0^{\sigma} \omega(s) ds \right)^2, \]
where \( \theta \) and \( \sigma \) are positive scalars, and \( \omega : [0, \sigma] \to \mathcal{R} \) is the function such that the integration is well defined, we have some bound of the terms on right-hand side of (10):
\[ -q_1 \int_{t-r_{11}}^{t} u^2(s) ds \leq - \frac{q_1}{\tau_{11}} \left( \int_{t-r_{11}}^{t} u(s) ds \right)^2 \]
\[ -q_2 \int_{t-r_{22}}^{t} v^2(s) ds \leq - \frac{q_2}{\tau_{22}} \left( \int_{t-r_{22}}^{t} v(s) ds \right)^2 \]
\[ -q_3 \int_{t-r_{12}}^{t} v^2(s) ds \leq - \frac{q_3}{\tau_{12}} \left( \int_{t-r_{12}}^{t} v(s) ds \right)^2 \]
\[ -q_4 \int_{t-r_{21}}^{t} u^2(s) ds \leq - \frac{q_4}{\tau_{21}} \left( \int_{t-r_{21}}^{t} u(s) ds \right)^2. \] (11)

Utilizing the inequality (11), we have a bound of \( \frac{dV}{dt} \) as
\[ \frac{dV}{dt} \leq 2(-a_{11} u(t) - a_{12} v(t)) p_1 \left( \frac{u(t)}{x^*} - a_{11} \int_{t-r_{11}}^{t} u(s) ds - a_{12} \int_{t-r_{12}}^{t} v(s) ds \right) \]
\[ + 2(-a_{21} u(t) - a_{22} v(t)) p_2 \left( \frac{v(t)}{y^*} - a_{21} \int_{t-r_{21}}^{t} u(s) ds - a_{22} \int_{t-r_{22}}^{t} v(s) ds \right) \]
\[ + \tau_{11} q_1 u^2(t) + \tau_{12} q_3 v^2(t) + \tau_{21} q_4 u^2(t) + \tau_{22} q_2 v^2(t) - \frac{q_1}{\tau_{11}} \left( \int_{t-r_{11}}^{t} u(s) ds \right)^2 \]
\[ - \frac{q_2}{\tau_{22}} \left( \int_{t-r_{22}}^{t} v(s) ds \right)^2 - \frac{q_3}{\tau_{12}} \left( \int_{t-r_{12}}^{t} v(s) ds \right)^2 - \frac{q_4}{\tau_{21}} \left( \int_{t-r_{21}}^{t} u(s) ds \right)^2, \]
\[ = \xi^T(t) \Sigma \xi(t) - \varepsilon_1 \left( \frac{u(t)}{x^*} \right)^2 - \varepsilon_2 \left( \frac{v(t)}{y^*} \right)^2 \] (12)

where
\[ \xi^T(t) = \left[ \frac{u(t)}{x^*} \frac{v(t)}{y^*} \int_{t-r_{11}}^{t} u(s) ds \int_{t-r_{12}}^{t} v(s) ds \int_{t-r_{21}}^{t} u(s) ds \int_{t-r_{22}}^{t} v(s) ds \right]. \]
Therefore, if the matrix $\Sigma$ is negative definite, there exist positive scalars $\gamma_1$ and $\gamma_2$ such that
\[
\frac{dV}{dt} \leq -\gamma_1 u^2(t) - \gamma_2 v^2(t).
\] (13)

Denote $\gamma = \min\{\gamma_1, \gamma_2\}$, then Eq. (13) leads to
\[
V(t) + \gamma \int_t^T [u^2(s) + v^2(s)]ds \leq V(T), \quad \text{for } t \geq T
\] (14)

and this implies $u^2(t) + v^2(t) \in L_1(T, \infty)$. It is straightforward to see from (4) and the boundedness of $Z(t)$ that $u^2(t) + v^2(t)$ is uniformly continuous. Then by Barbalat’s lemma [2], one can conclude that $\lim_{t \to \infty} [u^2(t) + v^2(t)] = 0$. Therefore, the zero solution of (4) is asymptotically stable. This completes the proof. \qed

**Corollary 2.** When $\tau_{12} = \tau_{21} = 0$ in system (4), the stability criterion of (7) is simplified to
\[
\begin{bmatrix}
-2x^*a_{11}p_1 + x^2\tau_{11}q_1 + \epsilon_1 & -y^*a_{12}p_1 - x^*a_{21}p_2 & x^*a_{11}^2p_1 & x^*a_{21}a_{22}p_2 \\
* & -2y^*a_{22}p_2 + y^2\tau_{22}q_2 + \epsilon_2 & y^*a_{12}a_{11}p_1 & y^*a_{22}^2p_2 \\
* & * & -q_1/\tau_{11} & 0 \\
* & * & * & -q_2/\tau_{22}
\end{bmatrix} < 0.
\] (15)

**Proof.** It can be easily obtained from the proof of Theorem 1, so it is omitted.

**Remark 3.** The LMI optimization problem (7) and (15) in Theorem 1 and Corollary 2 is to determine whether the problem is feasible or not. It is called the feasibility problem. The solutions of the problem can be found by solving the eigenvalue problem in $\mathbf{p}$ whether the problem is feasible or not. It is called the feasibility problem. The solutions of the problem

\[
\text{Remark 3. }
\] (15)

\[
\text{Proof. It can be easily obtained from the proof of Theorem 1, so it is omitted.}
\]

**Example 4.** Consider the system [5]
\[
\begin{align*}
\dot{x}(t) &= x(t)(1 - x(t - \tau_{11}) - 0.5y(t - \tau_{12})), \\
\dot{y}(t) &= y(t)(1 - 0.5x(t - \tau_{21}) - y(t - \tau_{22})).
\end{align*}
\] (16)

According to the result of Zhen and Ma [5], the positive equilibrium $(2/3, 2/3)$ of Eq. (16) is locally asymptotically stable if the following conditions hold:
\[
\begin{align*}
28\tau_{11} + 8\tau_{22} + 5\tau_{12} + 13\tau_{21} &< 18, \\
28\tau_{22} + 8\tau_{11} + 13\tau_{12} + 5\tau_{21} &< 18.
\end{align*}
\]

When $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = \tau$, the bound for local asymptotic stability of system (16) by the result in Zhen and Ma [5] is $0 < \tau < 0.34$. However, by solving the LMI (7), our criterion guarantees the stability for $0 < \tau < 0.88$. In particular, when $\tau_{12} = \tau_{11} = 0$, the stability bound of Zhen and Ma [5] is $0 < \tau_{11}, \tau_{22} < 0.5$, while the criterion (15) gives that $0 < \tau_{11}, \tau_{22} \leq 1.299$. When $\tau_{11} = \tau_{22} = 1.299$, the solutions of the LMI (15) are as follows:
\[
\begin{align*}
p_1 = p_2 = 197.5289, & \quad q_1 = q_2 = 171.0623, & \quad \epsilon_1 = \epsilon_2 = 0.0024.
\end{align*}
\]
References