Guaranteed cost control of time-delay chaotic systems

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Abstract

This article studies a guaranteed cost control problem for a class of time-delay chaotic systems. Attention is focused on the design of memory state feedback controllers such that the resulting closed-loop system is asymptotically stable and an adequate level of performance is also guaranteed. Using the Lyapunov method and LMI (linear matrix inequality) framework, two criteria for the existence of the controller are derived in terms of LMIs. A numerical example is given to illustrate the proposed method.

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1. Introduction

Since Lorenz found the first chaotic attractor in a simple three-dimensional autonomous systems in 1963, tremendous efforts have been devoted to chaos control including stabilization of unstable equilibria, and more generally, unstable periodic orbits during three decades. Also, since Mackey and Glass [1] first found chaos in time-delay system in 1977, there has been increasing attention in time-delay chaotic systems. For instance, see the papers; Lu and He [2], Tian and Gao [3], Chen and Yu [4], and the references therein. In the literature, one of the most frequent objectives consists in the stabilization of chaotic behaviors to one of unstable fixed points or unstable periodic orbits embedded within a chaotic attractor. That is, to design a stabilizing controller that guarantees the closed-loop system dynamics converges to the fixed point or periodic orbit. In this endeavor, Guan et al. [5], Sun [6], and Park and Kwon [7] have investigated the controller design problem of a class of time-delay chaotic systems using the famous Ott–Grebogi–Yorke (OGY) method [8]. They proposed two kind of controller, i.e., standard feedback control (SFC) and delayed feedback control (DFC), and derived the stabilization criteria using the Lyapunov method. For further information of DFC controller, see the paper [5]. In this article, we propose a novel feedback control scheme of integral type for the system.

On the other hand, when designing controllers for dynamic systems, it is desirable to ensure satisfactory system performance. One way to address this problem is so-called guaranteed cost control [9]. The approach has the advantage of providing an upper bound on a given performance index and thus the system performance degradation is guaranteed to

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be less than this bound. Thus, the guaranteed cost control problem for various dynamic systems have been investigated in recent years. For example, see the papers [10,11], and references therein. To the best of author's knowledge, the problem of guaranteed cost control for time-delayed chaotic systems has been overlooked to date. Thus we consider the guaranteed cost control problem for a class of time-delays chaotic systems in this paper. Using the Lyapunov stability analysis and LMI framework, two stabilization criteria for the system are derived. The solutions of the criteria can be easily obtained by various convex optimization algorithms.

**Notation:** $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote $n$-dimensional Euclidean space and the set of real $n$ by $m$ matrices, respectively. $\star$ denotes the symmetric part. $X > 0$ ($X \succ 0$) means that $X$ is a real symmetric positive definitive matrix (positive semi-definite). $I$ denotes the identity matrix with appropriate dimensions. $\| \cdot \|$ refers to the induced matrix 2-norm. $\text{diag}\{ \cdots \}$ denotes the block diagonal matrix. $\mathcal{C}_{a,b} = \mathcal{C}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the Banach space of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^n$, with the topology of uniform convergence.

### 2. Problem statement

Consider the following time-delay chaotic systems

$$
\dot{x}(t) = Ax(t) + Bx(t - h) + f_1(t, x(t)) + f_2(t, x(t - h)) + u(t),
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^n$ is the control input, $A, B \in \mathbb{R}^{n \times n}$ are constant system matrices representing the linear parts of the system, $f_1(t, x(t)), f_2(t, x(t - h)) \in \mathbb{R}^n$ are the nonlinear parts of the system, and $h > 0$ is the constant time delay.

Suppose that the chaotic system (1) has an unstable fixed point or an unstable periodic orbit $\bar{x}(t)$, and is currently in a chaotic state.

Then, the goal in this article is to control the system asymptotically converges towards $\bar{x}(t)$ via the following memory feedback:

$$
u(t) = -\mathcal{K} \left( (x(t) - \bar{x}(t)) + \int_{t-h}^{t} B(x(s) - \bar{x}(s)) \, ds \right)$$

(2)

in which $\mathcal{K}$ is a gain matrix of the controller and $\bar{x}(t)$ is the desired fixed point.

Since the chaotic system (1) has an unstable fixed point $\bar{x}(t) = \bar{x} = \text{constant}$, we have

$$
\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{x}(t - h) + f_1(t, \bar{x}(t)) + f_2(t, \bar{x}(t - h)).
$$

(3)

By applying the control input (2) to system (1), we obtain the following error dynamics:

$$
\dot{e}(t) = (A - \mathcal{K})e(t) + Be(t - h) + F_1(t, e(t)) + F_2(t, e(t - h)) - \mathcal{K} \int_{t-h}^{t} Be(s) \, ds,
$$

(4)

where $e(t) = x(t) - \bar{x}(t), e(t - h) = x(t - h) - \bar{x}(t - h) = x(t - h) - \bar{x}(t)$, and

$$
F_1(\cdot) = f_1(t, (e(t) + \bar{x}(t))) - f_1(t, \bar{x}(t)), \quad F_2(\cdot) = f_2(t, (e(t - h) + \bar{x}(t - h))) - f_2(t, \bar{x}(t - h)).
$$

Then, the control goal is to force $\| e(t) \| \rightarrow 0$ as $t \rightarrow \infty$.

### 3. Controller design

In this section, based on the Lyapunov stability theory and LMI framework, the design problem of guaranteed cost control for system (1) will be discussed.

For error system (4), since zero is a fixed point of $F_1(t, e(t)) + F_2(t, e(t - h))$, we have a Taylor expansion

$$
F_1(t, e(t)) + F_2(t, e(t - h)) = \beta_0 e(t) + [\text{HOT}]_1 + \beta_1 e(t - h) + [\text{HOT}]_2,
$$

(5)

where $\beta_0 = F_1'(t, e(t)), \beta_1 = F_2'(t, e(t - h)), [\text{HOT}]_1$ and $[\text{HOT}]_2$ are higher order term in $e(t)$ and $e(t - h)$, respectively, and $F_i'$ denotes the time derivative of $F_i$, $(i = 1, 2)$. 
From the OGY-method [8], the control goal is the zero fixed point, so we can only consider the linearized part near zero point. Rewrite the local error system as follows:

\[
\dot{e}(t) = (A - \mathcal{H} + \beta_0 I)e(t) + (B + \beta_1 I)e(t - h) - \mathcal{H}\int_{t-h}^t Be(s) \, ds.
\]  

(6)

Associated with the system (6) is the following quadratic cost function:

\[
J = \int_0^\infty (e^\top(t) \Gamma_1 e(t) + u^\top(t) \Gamma_2 u(t)) \, dt,
\]

(7)

where \( \Gamma_1 \in \mathbb{R}^{n\times n} \) and \( \Gamma_2 \in \mathbb{R}^{n\times n} \) are given positive-definite matrices.

Here, the objective of this article is to develop a procedure to design a state feedback controller \( u(t) \) for the system (4) and cost function (7) such that the resulting closed-loop system is asymptotically stable and the closed-loop value of the cost function (7) satisfies \( J \leq \mathcal{J} \), where \( \mathcal{J} \) is some specified constant.

**Definition 1.** For the error system (6) and cost function (7), if there exist a control law \( u^*(t) \) and a positive \( \mathcal{J} \) such that the system (6) is asymptotically stable and the closed-loop value of the cost function (7) satisfies \( J \leq \mathcal{J} \), then \( \mathcal{J} \) is said to be a guaranteed cost law of the system (6) and cost function (7).

Here, we introduce a well-known fact and two lemmas which are necessary for deriving main results of this paper.

**Fact 1 (Schur Complement).** Given constant symmetric matrices \( \Sigma_1, \Sigma_2, \Sigma_3 \) where \( \Sigma_1 = \Sigma_1^\top \) and \( 0 < \Sigma_2 = \Sigma_2^\top \), then \( \Sigma_1 + \Sigma_2 \Sigma_3^{-1} \Sigma_3 < 0 \) if and only if

\[
\begin{bmatrix}
\Sigma_1 & \Sigma_3 \\
\Sigma_3 & -\Sigma_2
\end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix}
-\Sigma_2 & \Sigma_3 \\
\Sigma_3^\top & -\Sigma_1
\end{bmatrix} < 0.
\]

**Lemma 1** [12]. For any constant matrix \( \Sigma \in \mathbb{R}^{n\times n}, \Sigma = \Sigma^\top > 0 \), scalar \( \gamma > 0 \), vector function \( \omega : [0, \gamma] \rightarrow \mathbb{R}^n \) such that the integrations concerned are well defined, then

\[
\left( \int_0^\gamma \omega(s) \, ds \right)^\top \Sigma \left( \int_0^\gamma \omega(s) \, ds \right) \leq \gamma \int_0^\gamma \omega^\top(s) \Sigma \omega(s) \, ds.
\]

(8)

**Lemma 2** [13]. Consider an operator \( \mathcal{D}(\cdot) : \mathcal{C}_{a,b} \rightarrow \mathbb{R}^n \) with \( \mathcal{D}(x(t)) = x(t) + \hat{B} \int_{t-h}^t x(s) \, ds \), where \( x(t) \in \mathbb{R}^n \) and \( \hat{B} \in \mathbb{R}^{n\times n} \). For a given scalar \( \delta \), where \( 0 < \delta < 1 \), if a positive definite symmetric matrix \( M \) exists, such that

\[
\begin{bmatrix}
-\delta M & h\hat{B}^\top \hat{M} \\
h\hat{M}\hat{B} & -\hat{M}
\end{bmatrix} < 0
\]

holds, then the operator \( \mathcal{D}(x(t)) \) is stable.

Now, define an operator \( \mathcal{D}(e_i) : \mathcal{C}_{a,b} \rightarrow \mathbb{R}^n \) as

\[
\mathcal{D}(e_i) = e(t) + \int_{t-h}^t Be(s) \, ds,
\]

(10)

where \( e_i = e(t + s), s \in [-h, 0] \).

From the definition of \( \mathcal{D}(e_i) \), the transformed error system is

\[
\dot{\mathcal{D}}(e_i) = \dot{e}(t) + Be(t) - Be(t - h) = (\bar{A} - \mathcal{H}) \mathcal{D}(e_i) + \beta_1 e(t - h) - \bar{A} \int_{t-h}^t Be(s) \, ds,
\]

(11)

where \( \bar{A} = A + B + \beta_1 J \).

This transformation is called parameterized neutral model transformation.

Then, we have the following theorem.

**Theorem 1.** For given \( \Gamma_1 \) and \( \Gamma_2 \), the system (1) under the control (2) is asymptotically stabilized to the given fixed point \( \bar{x}(t) \) if there exist positive definite matrices \( X, Z_1, Z_2, \) and any matrix \( Y \) which satisfy the following LMIs:
such that when \( \|e(t)\| \) is small enough, the control goal by the controller \( u(t) = -YX^{-1}(e(t) + \int_{t-h}^{t} Be(s) \, ds) \) is guaranteed, i.e.,
\[
\|e(t)\| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]

Also, the guaranteed cost of error system is
\[
\mathcal{J} = \mathcal{D}^T(0)P\mathcal{D}(0) + \int_{-h}^{t} (s + h)e^T(s)\mathcal{R}_1 e(s) \, ds + \int_{-h}^{t} e^T(s)\mathcal{R}_2 e(s) \, ds.
\]

**Proof.** For \( P > 0, \mathcal{R}_1 > 0, \) and \( \mathcal{R}_2 > 0 \), construct a Lyapunov functional of the form
\[
V = V_1 + V_2 + V_3,
\]
where
\[
V_1 = \mathcal{D}(e_i)^T P \mathcal{D}(e_i),
\]
\[
V_2 = \int_{t-h}^{t} (s - t + h)e^T(s)\mathcal{R}_1 e(s) \, ds,
\]
\[
V_3 = \int_{t-h}^{t} e^T(s)\mathcal{R}_2 e(s) \, ds.
\]

The differential of the Lyapunov functional along the trajectory of error system (11) is
\[
\frac{dV_1}{dt} = 2\mathcal{D}(e_i)^T P \left[ (\mathcal{A} - \mathcal{X})\mathcal{D}(e_i) - \mathcal{A} \int_{t-h}^{t} Be(s) \, ds + \beta_1 e(t - h) \right],
\]
\[
\frac{dV_2}{dt} = he^T(t)\mathcal{R}_1 e(t) - \int_{t-h}^{t} e^T(s)\mathcal{R}_1 e(s) \, ds \leq he^T(t)\mathcal{R}_1 e(t) - \left( \int_{t-h}^{t} e(s) \, ds \right)^T \left( \mathcal{R}_1^{-1} \right) \left( \int_{t-h}^{t} e(s) \, ds \right),
\]
\[
\frac{dV_3}{dt} = e^T(t)\mathcal{R}_2 e(t) - e^T(t - h)\mathcal{R}_2 e(t - h),
\]
where Lemma 1 are utilized in (20).

Define \( \mathcal{R} = h\mathcal{R}_1 + \mathcal{R}_2 \) and then
\[
e^T(t)\mathcal{R} e(t) = \mathcal{D}(e_i) - \int_{t-h}^{t} Be(s) \, ds \bigg) \mathcal{R} \left( \mathcal{D}(e_i) - \int_{t-h}^{t} Be(s) \, ds \right)
\]
\[
= \mathcal{D}^T(e_i)\mathcal{R} \mathcal{D}(e_i) - 2\mathcal{D}^T(e_i)\mathcal{R} \int_{t-h}^{t} Be(s) \, ds + \left( \int_{t-h}^{t} Be(s) \, ds \right)^T \mathcal{R} \left( \int_{t-h}^{t} Be(s) \, ds \right).
\]

From (19)–(21) and utilizing the relation (22), a new bound of the time-derivative of \( V \) is as follows:
\[
\frac{dV}{dt} = \sum_{i=1}^{3} \frac{dV_i}{dt} \leq \chi^T(t)\Omega\chi(t),
\]
Therefore, if

\[ \chi(t) = \left[ \frac{\mathcal{D}(e_t)}{\int_{t-h}^t e(s) \, ds} \right] \]

and

\[ \Omega = \begin{bmatrix} P(\mathcal{A} - \mathcal{X}) + (\mathcal{A} - \mathcal{X})^T P + \mathcal{R} & -P\mathcal{A}B - \mathcal{R}B & \beta_1 P \\ \mathcal{R} & -h^{-1} \mathcal{R}_1 + B^T \mathcal{R} & 0 \\ \mathcal{R} & \mathcal{R} & -\mathcal{R}_2 \end{bmatrix}. \]

Here, applying the relation (22) to the terms \( e^T(t) \Gamma_1 e(t) \) and using

\[ u^T(t) \Gamma_2 u(t) = \mathcal{D}(e_t) \mathcal{X}^T \Gamma_2 \mathcal{X} \mathcal{D}(e_t), \]

gives that

\[ \frac{dV}{dt} < \chi(t) \Omega \chi(t) - (e^T(t) \Gamma_1 e(t) + u^T(t) \Gamma_2 u(t)), \]

where

\[ \Omega_1 = \begin{bmatrix} P(\mathcal{A} - \mathcal{X}) + (\mathcal{A} - \mathcal{X})^T P + \mathcal{X}^T \Gamma_2 \mathcal{X} & -P\mathcal{A}B \beta_1 P \\ \mathcal{R} & -h^{-1} \mathcal{R}_1 & 0 \\ \mathcal{R} & \mathcal{R} & -\mathcal{R}_2 \end{bmatrix} + \begin{bmatrix} I \\ -B^T \mathcal{R} \mathcal{R}_1 & I \\ -B \mathcal{R}_1 \mathcal{R}_1 & -B \mathcal{R}_1 \mathcal{R}_1 \mathcal{R}_1 \mathcal{R}_1 \end{bmatrix} \]

Therefore, if \( \Omega_1 < 0 \), there exists the positive scalar \( \gamma \) such that

\[ \frac{dV}{dt} \leq -\gamma \||\mathcal{D}(e_t)||^2. \]

By Fact 1 (Schur Complement), the inequality \( \Omega_1 < 0 \) is equivalent to

\[ \Omega_2 = \begin{bmatrix} P(\mathcal{A} - \mathcal{X}) + (\mathcal{A} - \mathcal{X})^T P + \mathcal{X}^T \Gamma_2 \mathcal{X} & -P\mathcal{A}B \beta_1 P \\ \mathcal{R} & -h^{-1} \mathcal{R}_1 & 0 \\ \mathcal{R} & \mathcal{R} & -\mathcal{R}_2 \end{bmatrix} + \begin{bmatrix} I \\ -B^T \mathcal{R} \mathcal{R}_1 & I \\ -B \mathcal{R}_1 \mathcal{R}_1 & -B \mathcal{R}_1 \mathcal{R}_1 \mathcal{R}_1 \mathcal{R}_1 \end{bmatrix} < 0. \]

Letting \( X = P^{-1}, Y = \mathcal{X}^T X, Z_1 = h\mathcal{R}_1^{-1}, Z_2 = \mathcal{R}_2^{-1}, \) and pre- and post-multiplying the matrix \( \Omega_2 \) by diag \( \{X, Z_1, Z_2, I, I, \Gamma_1\} \), give that \( \Omega_2 < 0 \) is equivalent to the following inequality:

\[ \Omega_3 = \begin{bmatrix} \mathcal{A}X - Y + X\mathcal{A}^T - Y^T + Y^T \Gamma_2 Y & -\mathcal{A}Z_1 & \beta_1 \mathcal{Z}_2 & hX & X \Gamma_1 \\ \mathcal{R} & -Z_1 & 0 & -hZ_1 \mathcal{B}^T & -Z_1 \mathcal{B}^T & -Z_1 \mathcal{B}^T \Gamma_1 \\ \mathcal{R} & \mathcal{R} & -Z_1 & 0 & 0 \\ \mathcal{R} & \mathcal{R} & \mathcal{R} & -Z_1 & 0 \\ \mathcal{R} & \mathcal{R} & \mathcal{R} & \mathcal{R} & -Z_1 \end{bmatrix} < 0. \]

Again, by Fact 1, the fact that \( \Omega_3 < 0 \) is equivalent to the LMI (12).

The inequality (13) is equivalent to

\[ \begin{bmatrix} -P & hB^T P \\ \mathcal{R} & -P \end{bmatrix} < 0 \]

by pre- and post-multiplying the inequality (13) by diag \( \{X^{-1}, X^{-1}\} \). If the above inequality (31) holds, then we can prove that a positive scalar \( \delta \) which is less than one exists such that

\[ \begin{bmatrix} -\delta P & hB^T P \\ \mathcal{R} & -P \end{bmatrix} < 0 \]
according to matrix theory. Therefore, from Lemma 2, if the inequality (13) holds, then operator $D(e)$ is stable. This implies that both the original local error system (6) and transformed error system (11) with stable operator $D(e)$ are asymptotically stable by Theorem 9.8.1 in [16]. Furthermore, we have

$$e^T(t)\Gamma_1 e(t) + u^T(t)\Gamma_2 u(t) < -\frac{dV}{dt}.$$ 

Integrating both sides of the above inequality from 0 to $T_f$ leads to

$$\int_0^{T_f} (e^T(t)\Gamma_1 e(t) + u^T(t)\Gamma_2 u(t)) \, dt < V(0) - V(T_f) = (D^T(0)PD(0) - D^T(T_f)PD(T_f)) + \left( \int_{-h}^{0} (s+h)e^T(s)R_1 e(s) \, ds - \int_{T_f-h}^{T_f} (s+h)e^T(s)R_1 e(s) \, ds \right) + \left( \int_{-h}^{0} e^T(s)R_2 e(s) \, ds - \int_{T_f-h}^{T_f} e^T(s)R_2 e(s) \, ds \right),$$

where $D(0)$ denotes $D(e)|_{\tau=0}$.

As both the operator $D(e)$ and the system (6) are asymptotically stable, when $T_f \to \infty$,

$$D^T(T_f)PD(T_f) \to 0, \quad \int_{T_f-h}^{T_f} e^T(s)R_1 e(s) \, ds \to 0,$$

$$\int_{T_f-h}^{T_f} e^T(s)R_2 e(s) \, ds \to 0.$$

Therefore we have

$$\int_0^{\infty} (e^T(t)\Gamma_1 e(t) + u^T(t)\Gamma_2 u(t)) \, dt < V(0) = J.$$ (33)

This completes our proof. \(\square\)

Theorem 1 presents a method of designing a memory guaranteed cost feedback controller. The following theorem presents a method of selecting a controller minimizing the upper bound of the guaranteed cost (14).

Before stating our second result, let us define

$$\mathcal{N}_1, \mathcal{N}_1^T = \int_{-h}^{0} (s+h)e^T(s) e(s) ds, \quad \mathcal{N}_2, \mathcal{N}_2^T = \int_{-h}^{0} e^T(s) e(s) ds.$$ (34)

**Theorem 2.** Consider the system (6) with cost function (7). If the following optimization problem:

$$\min_{x, \Pi_1, \Pi_2, Z_1, Z_2, Y, z} \{ x + \text{Trace}(\Pi_1) + \text{Trace}(\Pi_2) \}$$

subject to

1. LMIs (12) and (13),
2. $\Pi_1 = \begin{bmatrix} -x & D^T(0) \\ D(0) & -X \end{bmatrix} < 0$,
3. $\Pi_2 = \begin{bmatrix} -\Pi_1 & h\mathcal{N}_1^{T} \\ h\mathcal{N}_1 & -hZ_1 \end{bmatrix} < 0$,
4. $\Pi_2 = \begin{bmatrix} -\Pi_2 & \mathcal{N}_2^{T} \\ \mathcal{N}_2 & -Z_2 \end{bmatrix} < 0$,

has a positive solution set $(X, \Pi_1, \Pi_2, Z_1, Z_2, Y, z)$, then the control law (2) is an optimal robust guaranteed cost control law which ensures the minimization of the guaranteed cost (14) for error system (6).
Proof. By Theorem 1, (i) in (35) is clear. Also, it follows from Fact 1 that (ii)–(iv) in (35) are equivalent to 
\( D'(0)X^{-1}S(0) < \alpha, h, \beta \) or \( \gamma < \Pi_1, \) and \( \gamma Z_1^{-1} \gamma < \Pi_2, \) respectively. On the other hand,
\[
\int_0^h (s + h)e^T(s)R_1e(s)ds = \int_0^h \text{Trace}((s + h)e^T(s)R_1e(s))ds = \text{Trace}(\gamma^T hZ^{-1} \gamma) < \text{Trace}(\Pi_1),
\]
\[
\int_0^h e^T(s)R_2e(s)ds = \int_0^h \text{Trace}(e^T(s)R_2e(s))ds = \text{Trace}(\gamma^T Z_1^{-1} \gamma) < \text{Trace}(\Pi_2).
\]
Hence, it follows from (14) that 
\[ J < \alpha + \text{Trace}(\Pi_1) + \text{Trace}(\Pi_2). \]

Thus, the minimization of \( \alpha + \text{Trace}(\Pi_1) + \text{Trace}(\Pi_2) \) implies the minimization of the guaranteed cost for the system (6). Note that this convex optimization problem guarantees that a global optimum, when it exists, is reachable (see Remark 1).

Remark 1. The problems in Theorems 1 and 2 are to determine whether the problem is feasible or not. It is called the feasibility problem. Also, the solutions of the problem can be found by solving eigenvalue problem with respect to solution variables, which is a convex optimization problem. For details of this kind of optimization problem, see Boyd et al. [14]. Various efficient convex optimization algorithms can be used to check whether the LMIs in Theorems 1 and 2 is feasible. In this article, in order to solve the LMIs, we utilize Matlab's LMI Control Toolbox [15], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [14].

4. Numerical example

Consider a chaotic system of the form [17,5–7]:
\[
\zeta \frac{dx(t)}{dt} = -x(t) + \frac{G}{1 + \mu} (x(t - h) + U_0) + \frac{U_M}{2} \cos(\pi x(t - h) + U_0 + U_M),
\]
where \( x(t) \) is the normalized output voltage variation, \( G \geq 0 \) is the feedback gain and \( h \geq 0 \) is the feedback delay, \( \mu \geq 0 \) is the fringe constant, \( \zeta \geq 0 \) is the response time, \( U_0 \) and \( U_M \) are the constant phase shifts, and \( U_M \) is the input phase shift induced by the fiber strain. Let us keep \( U_0 = 2.0, U_M = 1.0, U_M = -2.5 \) and put \( \mu = 1.0 \) and \( G = 2.0 \). When \( \zeta = 0.1 \) and \( h = 0.01 \), the system demonstrates chaotic behavior (for simulation results, see [5]). In this case, we know \( \bar{x} = 0.85 \) is one of the fixed points. Assume that the initial condition of the system \( x(t) = e^{-t} \) for \( t \in [-h, 0] \). Based on this fixed point, the feedback controller can be chosen as \( u(t) = X'((0.85 - x(t)) + \int_{t-h}^{t} (0.85 - x(s))ds) \). The error system is obtained that
\[
\dot{e}(t) = (-10 - X')e(t) + 10e(t - h) + F_1(t, e(t)) + F_2(t, e(t - h)) - 10X' \int_{t-h}^{t} e(s)ds,
\]
where
\[
F_1(\cdot) = 0, \quad \beta_0 = 0,
\]
\[
F_2(\cdot) = 10(e(t - h) + \bar{x}(t - h)) \cos(\pi e(t - h) + \bar{x}(t - h) - 0.5) + 10 \cos(\pi e(t - h) + \bar{x}(t - h) - 0.5) - 10 \cos(\pi \bar{x}(t - h) - 0.5).
\]
Then, we have
\[
\beta_1 = 10[\cos(\pi \bar{x}(t - h) - 0.5) - \pi(\bar{x}(t - h) + 1) \sin(\pi \bar{x}(t - h) - 0.5)]|_{\bar{x}(t-h)=0.85} = -53.6254.
\]
From the relationship in Eqs. (34) and (10), we have
\[
\Delta(0) = 1.1010, \quad \Delta_1 = 0.1229, \quad \Delta_2 = 0.1005.
\]
Now, for given $\Gamma_1 = 1$ and $\Gamma_2 = 0.1$, by applying the optimization problem given in Theorem 2 to the above system, we found that the LMI solutions of Theorem 2 are as follows:

\[
X = 0.0571, \quad Z_1 = 1.2922 \times 10^{-5}, \quad Z_2 = 0.0025, \quad Y = 10.0000, \quad \Pi_1 = 11.6853, \quad \Pi_2 = 0.0764, \quad \alpha = 21.2407.
\]

Thus, the gain of a stabilizing guaranteed cost controller is

\[
\mathcal{K} = YX^{-1} = 175.2219,
\]

and the upper bound of cost function is

\[
\mathcal{J} = \alpha + \Pi_1 + \Pi_2 = 37.0025.
\]

5. Concluding remarks

In this article, the design method of guaranteed control controller for stabilizing time-delay chaotic systems has been proposed. The delay-dependent stabilization criteria are derived in terms of LMIs by using the Lyapunov functional stability method. The criteria can be easily obtained by convex optimization algorithms. The design procedure is illustrated by a numerical example.

References