Delay-Dependent Guaranteed Cost Stabilization Criterion for Neutral Delay-Differential Systems: Matrix Inequality Approach

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Abstract—In this paper, we address the problem of the guaranteed cost stabilization for a class of neutral delay-differential systems with a given quadratic cost function. Based on the Lyapunov method, delay-dependent criteria, which are expressed in terms of matrix inequalities, are proposed to guarantee the asymptotic stability of the system. The matrix inequalities can be easily solved by various efficient optimization algorithms. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Neutral differential systems, Guaranteed cost control, Lyapunov method, Matrix inequalities.

NOMENCLATURE

\[ \mathbb{R}^n \] n-dimensional real space
\[ \mathbb{R}^{m \times n} \] set of all real m by n matrices
\[ A^T \] transpose of matrix A
\[ \| \cdot \| \] Euclidean vector norm or the induced matrix 2-norm
\[ P > 0 \] matrix P is symmetric positive definite
\[ A \geq B \] matrix A - B is symmetric positive semidefinite
\[ I \] identity matrix with appropriate dimension

**1. INTRODUCTION**

The purpose of this paper is to investigate the stabilization problem of a class of linear functional differential systems of neutral type

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + C \dot{x}(t-r) + D \int_{t-H}^{t} x(s) \, ds + Bu(t), \quad t \geq 0,
\]

\[
x(t \geq 0 + \theta) = \phi(\theta), \quad \forall \theta \in [-H, 0],
\]

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where \(x(t) \in \mathbb{R}^n\) is the state vector, \(A_0, A_1, C,\) and \(D \in \mathbb{R}^{n \times n}\) are constant matrices, \(B \in \mathbb{R}^{n \times m}\), \(u(t) \in \mathbb{R}^m\) is a control vector, \(h\) and \(\tau\) are positive constant time-delays, \(H = \max\{h, \tau, \eta\}\), \(\phi(.)\) is the given continuously differentiable function on \([-H,0]\), and it is assumed that the pair \((A_0 + A_1 + \eta D, B)\) is completely controllable.

As is well known, since time delays are frequently encountered in mathematics, engineering, biology, economy, population dynamics, and other areas, and represent a source of instability and poor performance of the systems, the study of delay-differential systems has received considerable attention over the past decades (see [1-3]). In recent years, there have been a number of studies on stability analysis of neutral differential system of the forms (see [4-13])

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - h) + C \phi(t - \tau),
\]

\[
\dot{x}(t) = A x(t) + \sum_{i=1}^{m} [A_{1i} x(t - h_i) + C_i \phi(t - \tau_i)],
\]

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - h) + C \phi(t - \tau) + D \int_{t-\eta}^{t} x(s) \, ds.
\]

There is a common goal in the above works, which is to present a less conservative condition to guarantee asymptotic stability of the systems. On the other hand, the problem of controller design for stabilization of such neutral systems with control input has been studied by only a few researchers (see [14-17]). Moreover, no further design procedure is considered to select a particular controller amongst all the admissible stabilizing controllers [14-17]. One way to address the performance problem for dynamic systems is to consider a linear quadratic cost function. This approach is the so-called guaranteed cost control [18-20]. The approach has the advantage of providing an upper bound on a given performance index, and thus, the system performance degradation incurred by delays is guaranteed to be less than this bound. Unfortunately, to the best of the author’s knowledge, none of the known guaranteed cost controls can be applied to system (1). Hence, the objective of this article is to develop a procedure to design the state feedback controller for system (1) with a given cost function, such that the closed-loop system is asymptotically stable and the closed-loop value of the cost function is not more than a specified upper bound.

In this paper, associated with system (1) is the following quadratic cost function

\[
J = \int_{0}^{\infty} (x^{T}(t)Q x(t) + u^{T}(t)S u(t)) \, dt,
\]

where \(Q \in \mathbb{R}^{n \times n}\) and \(S \in \mathbb{R}^{m \times m}\) are given positive-definite matrices.

In general, abandonment of information on the delay causes conservativeness of the stability criteria especially when delays are small. Delay-dependent criteria are often less conservative than delay-independent criteria. In this paper, based on the Lyapunov functional approach combined with a matrix inequality technique, several delay-dependent criteria for the existence of the guaranteed cost controller are derived in terms of matrix inequalities. The inequalities can be easily solved by various efficient convex optimization algorithms [21]. Finally, two numerical examples are given to illustrate the proposed method.

2. MAIN RESULTS

System (5) can be rewritten in the following form:

\[
\frac{d}{dt} \left[ x(t) + A_1 \int_{t-h}^{t} x(s) \, ds - C \phi(t - \tau) + D \int_{t-\eta}^{t} (s - t + \eta) x(s) \, ds \right] = \left( A_0 + A_1 + \eta D \right) x(t) + B u(t) \equiv A x(t) + B u(t), \quad t \geq 0,
\]

where \(A = A_0 + A_1 + \eta D\).
Define the operator $\mathcal{D} : C([-H, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ as
\[
\mathcal{D}(x_t) = x(t) + A_1 \int_{t-h}^t x(s) ds - Cx(t - \tau) + D \int_{t-\eta}^t (s - t + \eta)x(s) ds.
\]
Note that a sufficient condition for stability of the operator $\mathcal{D}$ is
\[
h \|A_1\| + \|C\| + (\eta^2/2)\|D\| < 1 \quad [4].
\]
Now, we synthesize a memoryless state feedback controller of the form
\[
u(t) = -B^T P x(t), \quad (4)
\]
where $P$ is a positive-definite matrix to be designed later.

By applying the control law (4), the closed-loop system of system (1) is given by
\[
\frac{d}{dt} [\mathcal{D}(x_t)] = (A - BB^T P) x(t). \quad (5)
\]

We will present here a definition and lemmas which will be used in the rest of the paper.

**DEFINITION 2.1.** For neutral delay system (1) and cost function (2), if there exist a control law $u^*(t)$ and a positive scalar $J^*$ such that the closed-loop system (5) is asymptotically stable and the closed-loop value of the cost function (2) satisfies $J \leq J^*$, then $J^*$ is said to be a guaranteed cost and $u^*(t)$ is said to be a guaranteed cost control law of system (1) and the cost function (2).

**LEMMA 2.2.** (See [23].) For any constant symmetric positive-definite matrix $M$, a scalar $\sigma > 0$, and the vector function $\omega : [0, \sigma] \rightarrow \mathbb{R}^m$ such that the integrations in the following are well defined, then
\[
\sigma \int_0^\sigma \omega^T(s)M\omega(s) ds \geq \left( \int_0^\sigma \omega(s) ds \right)^T M \left( \int_0^\sigma \omega(s) ds \right).
\]

**LEMMA 2.3.** (See [21].) The following matrix inequality
\[
\begin{bmatrix}
Q_1(x) & Q_2(x) \\
Q_3(x) & Q_3(x)
\end{bmatrix} > 0,
\]
where $Q_1(x) = Q_1^T(x)$, $Q_3(x) = Q_3^T(x)$, and $Q_2(x)$ depend affinely on $x$, is equivalent to
\[
Q_3(x) > 0 \quad \text{and} \quad Q_1(x) - Q_2(x)Q_3(x)^{-1}Q_2^T(x) > 0.
\]

The following are our main results for delay-dependent criteria for asymptotic stabilization of system (1).

**THEOREM 2.4.** For given $h > 0$, $\tau > 0$, $\eta > 0$, $Q > 0$, and $S > 0$, suppose that the operator $\mathcal{D}$ is stable and there exist a positive scalar $\alpha$ and $X > 0$ satisfying the following matrix inequality:
\[
\begin{bmatrix}
XA^T + AX - 2BB^T + BSBS^T & \sqrt{\frac{2}{3}}\eta^2XD^T & \eta XA^T & h^{1/2}XA_1^T \\
* & -\eta X & 0 & 0 \\
* & * & -\eta X & 0 \\
* & * & * & -X \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\alpha XQ^T & QX & -XA^T + BB^T & hXA^T - hBB^T
\end{bmatrix} < 0,
\]
\[
(6)
\]
where $X = P^{-1}$. Then, system (1) is asymptotically stable by a guaranteed cost control law $u(t) = -B^T P x(t)$ and the quadratic cost function $J$ satisfies the bound

$$
J^* = D^T(0)PD(0) + \int_{-\tau}^{0} (s + h) x^T(s)A_1^T PA_1 x(s) ds
+ \alpha \int_{-\tau}^{0} x^T(s)C^T PC x(s) ds + \frac{2}{3} \int_{-\eta}^{0} (s + \eta)^3 x^T(s)D^T PD x(s) ds.
$$

PROOF. We employ the following legitimate Lyapunov functional candidate [2]:

$$
V = D^T(x_t)PD(x_t) + \int_{t-h}^{t} (s - t + h) x^T(s)A_1^T PA_1 x(s) ds
+ \alpha \int_{t-\tau}^{t} x^T(s)C^T PC x(s) ds + \frac{2}{3} \int_{t-\eta}^{t} (s - t + \eta)^3 x^T(s)D^T PD x(s) ds.
$$

Taking the time derivative of $V$ along the solution of (3), we have

$$
\frac{dV}{dt} = 2x^T(t) \left( A^T - BB^T P \right) P \left( x(t) + A_1 \int_{t-h}^{t} x(s) ds - Cx(t - \tau) \right)
+ D \int_{t-\eta}^{t} (s - t + \eta) x(s) ds + hx^T(t)A_1^T PA_1 x(t)
- \int_{t-h}^{t} x^T(s)A_1^T PA_1 x(s) ds + \alpha x^T(t)C^T PC x(t)
- \alpha x^T(t - \tau)C^T PC x(t - \tau) + \frac{2}{3} \eta^3 x^T(t)D^T PD x(t)
- 2 \int_{t-\eta}^{t} (s - t + \eta)^2 x^T(s)D^T PD x(s) ds.
$$

Using Lemma 2.2 and the well-known inequality $2a^T b \leq \varepsilon a^T H a + \varepsilon^{-1} b^T H^{-1} b$, where $\varepsilon$ is a positive scalar, $H > 0$, and $a$ and $b$ are vectors with appropriate dimension, we can obtain a bound of two terms in (9) as

$$
2x^T(t)A^T PD \int_{t-\eta}^{t} (s - t + \eta) x(s) ds \leq \eta x^T(t) A^T PA x(t)
+ \eta^{-1} \left( PD \int_{t-\eta}^{t} (s - t + \eta) x(s) ds \right)^T P^{-1} \left( PD \int_{t-\eta}^{t} (s - t + \eta) x(s) ds \right)
\leq \eta x^T(t) A^T PA x(t) + \int_{t-\eta}^{t} (s - t + \eta)^2 x^T(s)D^T PD x(s) ds,
$$

$$
- \int_{t-h}^{t} x^T(s)A_1^T PA_1 x(s) ds \leq - \left( \frac{1}{h} P A_1 \int_{t-h}^{t} x(s) ds \right)^T \left( \frac{1}{h} P^{-1} \right) \left( \frac{1}{h} P A_1 \int_{t-h}^{t} x(s) ds \right).
$$

Then, we obtain

$$
\frac{dV}{dt} \leq x^T(t) \left( A^T P + PA + \eta A^T PA - 2BB^T P + hA_1^T PA_1 + \alpha C^T PC + \frac{2}{3} \eta^3 D^T PD \right)
\times x(t) - \alpha x^T(t - \tau)C^T PC x(t - \tau) - 2x^T(t)A^T PC x(t - \tau)
- \left( \frac{1}{h} P A_1 \int_{t-h}^{t} x(s) ds \right)^T \left( \frac{1}{h} P^{-1} \right) \left( \frac{1}{h} P A_1 \int_{t-h}^{t} x(s) ds \right).
$$
\[ +2x^T(t)A^TPA_1 \int_{t-h}^t x(s)\,ds - 2x^T(t)PBB^TPA_1 \int_{t-h}^t x(s)\,ds \]
\[ +2x^T(t)PBB^TPC(x(t-\tau) - 2x^T(t)PBB^TPD \int_{t-\eta}^t (s - t + \eta)x(s)\,ds \] (12) (cont.)
\[ - \int_{t-\eta}^t (s - t + \eta)^2 x^T(s)D^TPDx(s)\,ds. \]

Again using Lemma 2.2, we get a bound of the last term of the right-hand side in (12)

\[ - \int_{t-\eta}^t (s - t + \eta)^2 x^T(s)D^TPDx(s)\,ds \leq - \left( \frac{1}{\eta} PD \int_{t-\eta}^t (s - t + \eta)x(s)\,ds \right)^T \]
\[ \cdot (\eta P^{-1}) \left( \frac{1}{\eta} PD \int_{t-\eta}^t (s - t + \eta)x(s)\,ds \right). \]

Thus, we have
\[ \frac{dV}{dt} \leq \xi^T(t)\Sigma(t) - x^T(t) (Q + PBSB^T P) x(t), \] (14)

where
\[ \xi(t) = \left[ x^T(t) x^T(t-\tau) C^TP \left( \frac{1}{h} PA_1 \int_{t-h}^t x(s)\,ds \right)^T \left( \frac{1}{\eta} PD \int_{t-\eta}^t (s - t + \eta)x(s)\,ds \right)^T \right]^T \]
and
\[ \Sigma = \begin{bmatrix} (1,1) & -A^T + PBB^T & hA^T - hPBB^T & -\eta PBB^T \\ * & -\alpha P^{-1} & 0 & 0 \\ * & * & -hP^{-1} & 0 \\ * & * & * & -\eta P^{-1} \end{bmatrix}, \] (15)

where \((1,1) = A^TP + PA - 2PBB^TP + hA_1^TPA_1 + \alpha C^TPC + (2/3)\eta^3 D^TPD + \eta A^TPA + Q + PBSB^TP.\)

Hence, if \(\Sigma < 0\), there exists a positive scalar \(\gamma\) such that \(\frac{dV}{dt} < -\gamma \|x(t)\|^2\), where \(\gamma = \lambda_m(Q + PBSB^TP)\). Noting that the operator \(\mathcal{D}\) is stable, system (1) is asymptotically stable according to Theorems 9.3.5 and 9.8.1 of [2].

By premultiplying and postmultiplying matrix \(\Sigma\) by matrix diag\{X, I, I, I\}, the fact that \(\Sigma < 0\) is equivalent to
\[ \left[ \begin{array}{cccc} XA^T + AX - 2BB^T \\ +hXA_1^TX^{-1}A_1X \\ +\alpha XC^TX^{-1}CX \\ +\frac{2}{3} \eta^3 XD^TX^{-1}DX \\ +\eta XA^T X^{-1}AX \\ +XQX + BSB^T \end{array} \right] \begin{bmatrix} -XA^T + BB^T & hXA^T - hBB^T & -\eta BB^T \\ -\alpha X & 0 & 0 \\ -hX & 0 \\ -\eta X \end{bmatrix} < 0. \] (16)

By Lemma 2.3, inequality (16) is equivalent to the matrix inequality (6). Also, the matrix inequality (6) implies that
\[ \frac{dV}{dt} < -x^T(t) (Q + PBSB^T P) x(t) < 0. \] (17)
Integrating both sides of the above inequality from 0 to \(T_f\) leads to

\[
\int_{0}^{T_f} x^T(t) \left( Q + PBSB^T P \right) x(t) \, dt < V(0) - V(T_f) = (D^T(0) PD(0) - D^T(T_f) PD(T_f)) \\
+ \left( \int_{-h}^{0} (s + h)x^T(s)A_1^T PA_1 x(s) \, ds - \int_{T_f-h}^{T_f} (s - T_f + h)x^T(s)A_1^T PA_1 x(s) \, ds \right) \\
+ \left( \alpha \int_{-\tau}^{0} x^T(s)C^T PCx(s) \, ds - \alpha \int_{T_f-\tau}^{T_f} x^T(s)C^T PCx(s) \, ds \right) \\
+ \frac{2}{3} \left( \int_{-\eta}^{0} (s + \eta)^3 x^T(s)D^T PDx(s) \, ds \int_{T_f-\eta}^{T_f} (s - T_f + \eta)^3 x^T(s)D^T PDx(s) \, ds \right).
\]

As the closed-loop system (5) is asymptotically stable, when \(T_f \to \infty\),

\[
D^T(T_f) PD(T_f) \to 0, \quad \int_{T_f-h}^{T_f} (s - T_f + h)x^T(s)A_1^T PA_1 x(s) \, ds \to 0, \\
\alpha \int_{T_f-\tau}^{T_f} x^T(s)C^T PCx(s) \, ds \to 0, \quad \int_{T_f-\eta}^{T_f} (s - T_f + \eta)^3 x^T(s)D^T PDx(s) \, ds \to 0.
\]

Hence, we get

\[
\int_{0}^{\infty} x^T(t) \left( Q + PBSB^T P \right) x(t) \, dt \leq V(0) = J^*.
\]

This completes the proof. \(\blacksquare\)

**Remark 2.5.** The solutions of the feasibility problem (6) can be found by solving a generalized eigenvalue problem in \(X\) and \(\alpha\), which is a quasi-convex optimization problem. Note that a locally optimal point of a quasi-convex optimization problem with strictly quasi-convex objective is globally optimal. For details, see [21]. Various efficient convex optimization algorithms can be used to check whether the matrix inequality (6) is feasible.

Theorem 2.4 presents a method of designing a state feedback guaranteed cost controller. The following theorem presents a method of selecting a controller minimizing the upper bound of the guaranteed cost (7).

**Theorem 2.6.** Consider system (5) with cost function (2). If the following optimization problem

\[
\min_{X>0, \alpha>0, \beta>0, \Gamma_1>0, \Gamma_2>0, \Gamma_3>0} \{ \beta + \text{tr} (\Gamma_1) + \text{tr} (\Gamma_2) + \text{tr} (\Gamma_3) \} \\
\text{subject to} \\
\begin{cases}
(i) \text{matrix inequality (6)} \\
(ii) \left[ \begin{array}{cc}
-\beta & D^T(0) \\
D(0) & -X
\end{array} \right] < 0, \\
(iii) \left[ \begin{array}{cc}
-\Gamma_1 & \alpha N_1^T C^T \\
\alpha C N_1 & -\alpha X
\end{array} \right] < 0, \\
(iv) \left[ \begin{array}{cc}
-\Gamma_2 & N_2^T A_1^T \\
A_1 N_2 & -X
\end{array} \right] < 0, \\
(v) \left[ \begin{array}{cc}
-\Gamma_3 & N_3^T D^T \\
D N_3 & -X
\end{array} \right] < 0,
\end{cases}
\]

has a positive solution set \((X, \alpha, \beta, \Gamma_1, \Gamma_2, \Gamma_3)\), then the control law (4) is an optimal guaranteed cost control law which ensures the minimization of the guaranteed cost (18) for the neutral
system (5), where \( \int_{-\tau}^{0} x(s)x^T(s)\, ds = N_1N_1^T, \) \( \int_{-h}^{0} (s + h)x(s)x^T(s)\, ds = N_2N_2^T, \) and \( (2/3) \cdot \int_{-\eta}^{0} (s + \eta)^3 x(s)x^T(s)\, ds = N_3N_3^T. \)

**Proof.** By Theorem 2.4, (i) in (19) is clear. Also, it follows from Lemma 2.3 that (ii), (iii), and (iv) in (19) are equivalent to \( D^T(0)X^{-1}D(0) < \beta, \alpha N_1^T C^T PCN_1 < \Gamma_1, N_2^T A_1^T PA_1 N_2 < \Gamma_2, \) and \( N_3^T D^T P D N_3 < \Gamma_3, \) respectively. On the other hand,

\[
\alpha \int_{-\tau}^{0} x^T(s)PCx(s)\, ds = \alpha \int_{-\tau}^{0} \text{tr} \left( x^T(s)PCx(s) \right)\, ds = \text{tr} \left( \alpha C^T PC \int_{-\tau}^{0} x(s)x^T(s)\, ds \right) = \text{tr} \left( \alpha C^T PCN_1N_1^T \right) = \text{tr} \left( \alpha N_1^T C^T PCN_1 \right) < \text{tr} (\Gamma_1).
\]

Similarly,

\[
\int_{-h}^{0} (s + h)x^T(s)A_1^T PA_1 x(s)\, ds = \text{tr} \left( A_1^T PA_1 N_2^T N_2 \right) = \text{tr} \left( N_2^T A_1^T PA_1 N_2 \right) < \text{tr} (\Gamma_2),
\]

\[
\left( \frac{2}{3} \right) \cdot \int_{-\eta}^{0} (s + \eta)^3 x^T(s)D^T P D x(s)\, ds = \text{tr} \left( D^T P D N_3^T N_3 \right) = \text{tr} \left( N_3^T D^T P D N_3 \right) < \text{tr} (\Gamma_3).
\]

Hence, it follows from (7) that

\[
J^* < \beta + \text{tr} (\Gamma_1) + \text{tr} (\Gamma_2) + \text{tr} (\Gamma_3).
\]

Thus, the minimization of \( \beta + \text{tr} (\Gamma_1) + \text{tr} (\Gamma_2) + \text{tr} (\Gamma_3) \) implies the minimization of the guaranteed cost for system (5). In light of Remark 2.5, the convexity of this optimization problem ensures that a global optimum, when it exists, is reachable.

**Remark 2.7.** In this paper, in order to solve the matrix inequalities (19), we utilize MATLAB LMI Control Toolbox [22], which implements state-of-the-art interior-point algorithms and is significantly faster than classical convex optimization algorithms [21].

In order to illustrate the design procedure of the proposed method, we will use the following two numerical examples.

**Example 2.8.** Consider the following neutral system:

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - 0.3) + C \dot{x}(t - 0.2) + D \int_{t-0.5}^{t} x(s)\, ds + B u(t), \tag{20}
\]

where

\[
A_0 = \begin{bmatrix} -1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0.5 \\ 0.5 & 0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},
\]

and the initial condition of the system is as follows:

\[
x(t) = [x_1(t) x_2(t)]^T = [1 - 1]^T, \quad \text{for} \quad -0.5 \leq t \leq 0.
\]
Here, we are about to construct a memoryless state feedback controller of form (4) for system (20) such that the closed-loop system is asymptotically stable and a corresponding upper bound of the cost function
\[ J = \int_0^\infty \left( x_1^2(t) + x_2^2(t) + 0.2u^2(t) \right) dt \]
is minimized. That is, \( Q = I \) and \( S = 0.2I \).

From the relations \( D(0) = x(0) + A_1 \int_{-0.3}^0 x(s) ds - Cx(-0.2) + D \int_{-0.5}^0 (s+\eta)x(s) ds \), \( N_1 N_1^T = \int_{-0.1}^0 x(s)x^T(s) ds \), \( N_2 N_2^T = \int_{-0.5}^0 (s+\eta)x(s)x^T(s) ds \), \( N_3 N_3^T = (2/3) \int_{-\eta}^0 (s+\eta)^2 x(s)x^T(s) ds \), we have
\[
D(0) = \begin{bmatrix} 0.8650 \\ -0.6150 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0.3162 & -0.3162 \\ -0.3162 & 0.3162 \end{bmatrix},
\]
\[
N_2 = \begin{bmatrix} 0.1500 & -0.1500 \\ -0.1500 & 0.1500 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0.0722 & -0.0722 \\ -0.0722 & 0.0722 \end{bmatrix}.
\]

By solving the optimization problem of Theorem 2.6 using the software LMI Toolbox in MATLAB [22], it is found that the problem is feasible, and the solutions are
\[
X = \begin{bmatrix} 0.7098 & -0.0955 \\ -0.0955 & 0.4460 \end{bmatrix}, \quad \alpha = 0.0284, \quad \beta = 1.9284,
\]
\[
\Gamma_1 = 10^{-5} \times \begin{bmatrix} 0.1365 & 0.0194 \\ 0.0194 & 0.1365 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0.0051 & -0.0051 \\ -0.0051 & 0.0051 \end{bmatrix},
\]
\[
\Gamma_3 = \begin{bmatrix} 0.0073 & -0.0073 \\ -0.0073 & 0.0073 \end{bmatrix}.
\]

Thus, the stabilizing optimal guaranteed cost control law, \( u(t) \), for system (20) is given by
\[
u(t) = -B^TPx(t) = -B^TX^{-1}x(t) = -[1.0358 \ 2.4640]x(t),
\]
and the corresponding optimal guaranteed cost of the closed-loop system is
\[ J^* = \alpha + \text{tr} (\Gamma_1) + \text{tr} (\Gamma_2) + \text{tr} (\Gamma_3) = 1.949. \]

**Example 2.9.** Consider the following scalar neutral delay-differential system:
\[
\dot{x}(t) = 0.1x(t) + 0.2x(t-0.5) + 0.2x(t-\tau) + 0.8 \int_{t-0.5}^t x(s) ds + u(t), \quad t \geq 0, \quad (21)
\]
\[
x(t) = e^t, \quad t \in [-\tau, 0],
\]
where \( \tau \geq 0.5 \). Comparing (21) with (1), we have \( A = 0.1, A_1 = 0.2, C = 0.2, D = 0.8, \) and \( h = \eta = 0.5 \). Note that the system is unstable when the control input \( u(t) \) is not forced, \( u(t) = 0 \). Here, by applying Theorem 2.6, we are about to find the maximum bound of \( \tau \) for guaranteeing stability. Consider the cost function (2) with \( Q = 1 \) and \( S = 0.2 \). First, by simple calculations, we can get
\[ D(0) = 0.8 + 0.6e^{-0.5} - 0.2e^{-\tau}, \quad N_1 = \frac{1}{2} \left( 1 - e^{-2\tau} \right), \quad N_2 = 0.3033, \quad N_3 = 0.1742.
\]
By iteratively solving the LMIs (19) for increasing \( \tau \), we can find that the allowable maximum bound for guaranteeing stability is \( \tau = \infty \). As \( \tau \) increases, the stabilizing controller law and corresponding optimal cost are approached to
\[ u(t) = -2.0059x(t) \quad \text{and} \quad J^* = 2.7322. \]

This implies that system (21) is asymptotically stable independent of delay \( \tau \) by the control law (22).
In this article, we presented a solution to the optimal guaranteed cost control problem via memoryless state feedback control laws for a class of neutral delay systems. Based on the Lyapunov method, delay-dependent criteria for asymptotic stability of the system were proposed in terms of matrix inequalities which can be easily solved by various optimization algorithms. Two numerical examples were used to illustrate the design procedure of the controller.

REFERENCES