The problem of the stability analysis for uncertain linear neutral systems with time-varying delay is investigated. Using Lyapunov method with linear matrix inequality (LMI), a new delay-independent criterion for robust stability of the systems is presented. A numerical example is given to illustrate the application of the proposed method.

**Keywords:** Neutral system; Robust stability; Lyapunov method; LMI

1. **INTRODUCTION**

It is commonly recognized that time-delays are natural components of dynamic processes (chemical or electrical) or biological systems involving propagation and transport phenomena or heredity, and frequently a source of instability ([3,5,11]). This is why the study of dynamic systems with time-delays has received considerable attention over the past years. Especially, in the recent years, the stability analysis of neutral delay-differential systems has received some attention. In the literatures ([2,6–10,12–16]), the stability problem of the neutral systems has been extensively investigated using various techniques such as Lyapunov method, characteristic equation approach, or state solution approach, and a number of stability criteria have been presented. However, most of the stability criteria in the literature are expressed in terms of matrix norms or matrix measures of the system matrices. Unfortunately, the matrix measure and matrix norm operations usually make the criteria more conservative. Furthermore, the work can be applicable only to the system with time-invariant delays, and did not consider the system uncertainties.

In this article, the problem of the stability analysis for a class of linear neutral systems with uncertainties and time-varying delay is investigated. Using Lyapunov method, a new delay-independent criterion for robust stability of the system is derived.
The criterion is expressed in terms of a linear matrix inequality (LMI) to make the stability criterion less conservative. The solutions of the LMI can be found easily by various effective optimization algorithms [1]. Finally, a numerical example is given to illustrate the application of the proposed method.

Throughout this article, the following notations are used. $\mathbb{R}^+$ is the set of nonnegative real numbers, $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space, and $\mathbb{R}^{n \times m}$ denotes the set of all $n \times m$ real matrices. $I$ denotes identity matrix of appropriate order, $*$ denotes symmetric terms, \text{diag}\{\cdots\} denotes the block diagonal matrix, and $\rho(A)$ denotes the largest modulus of the eigenvalues of the matrix $A$, which is known as the spectral radius of $A$. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote minimum and maximum eigenvalue of $A$, respectively. $C_r = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The following norms will be used: $\| \cdot \|$ denotes the Euclidean vector norm or the induced matrix 2-norm; $\| \phi \|_c = \sup_{-\tau \leq t \leq 0} \| \phi(t) \|$ stands for the norm of a function $\phi \in C_r$. Moreover, we denote by $C_r^u = \{ \phi \in C_r; \| \phi \|_c < u \}$, where $u$ is a positive real number. For any square matrices $X$ and $Y$, the notation $X \succeq Y$ (respectively, $X \succ Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite).

2. MAIN RESULT

Consider the following linear neutral systems with uncertainties and time-varying delay:

$$
\begin{align*}
\dot{x}(t) - C \dot{x}(t - h(t)) &= (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t - h(t)), \\
x(t_0 + \theta) &= \phi(\theta), \quad \theta \in [-\tilde{h}, 0], \quad (t_0, \phi) \in \mathbb{R}^+ \times C_r^u \quad (1)
\end{align*}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $h(t)$ is time-varying bounded delay satisfying

$$
0 < h(t) \leq \tilde{h} < \infty, \quad \dot{h}(t) \leq \tilde{h}_d < 1, \quad (2)
$$

$\phi(\cdot)$ is a given function specifying the initial condition, $A$, $B$ and $C \in \mathbb{R}^{n \times n}$ are constant system matrices, and $\Delta A(\cdot)$ and $\Delta B(\cdot)$ are unknown real norm-bounded matrix functions which represent time-varying parameter uncertainties. The admissible uncertainties are assumed to be of the form

$$
\Delta A(t) = D_a F_a(t) E_a, \quad \Delta B(t) = D_b F_b(t) E_b \quad (3)
$$

where $D_a, D_b, E_a$ and $E_b$ are real constant matrices with appropriate dimensions, and $F_a \in \mathbb{R}^{k_a \times l_a}$ and $F_b(t) \in \mathbb{R}^{k_b \times l_b}$ are unknown matrices satisfying

$$
\| F_a(t) \| \leq 1, \quad \| F_b(t) \| \leq 1.
$$
For the basic theory of neutral delay-differential systems, the reader is referred to the books by Hale and Verduyn Lunel [5].

Then, the goal of this article is to find the criterion for robust stability of the system (1).

Before proceeding to the main result, the following lemma is provided.

**Lemma 1** [1] The linear matrix inequality

\[
\begin{bmatrix}
Q(x) & S(x) \\
S(x)^T & R(x)
\end{bmatrix} > 0,
\]

is equivalent to

\[
R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^T > 0,
\]

where \( Q(x) = Q(x)^T, \ R(x) = R(x)^T \) and \( S(x) \) depend affinely on \( x \).

Now, we establish a delay-independent criterion, for robust stability of the neutral delay-differential system (1) using Lyapunov method with LMI.

**Theorem 1** For given scalar \( \tilde{h}_d \) satisfying (2), the system given in (1) is robustly stable independent of delay if \( \rho(C) < 1 \) and there exist the positive scalars \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) and the positive definite matrices \( P \) and \( R \) satisfying the following inequality:

\[
\Sigma(P, R, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) =
\begin{bmatrix}
\Sigma_{11} & \varepsilon_1 PD_a & \varepsilon_3 PD_a & PB - A^T PC & 0 & 0 & \tilde{E}^T \\
* & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\
* & * & -\varepsilon_3 I & 0 & 0 & 0 & 0 \\
* & * & * & \Sigma_{44} & \varepsilon_2 C^T PD_b & \varepsilon_4 C^T PD_b & 0 \\
* & * & * & * & -\varepsilon_1 I & 0 & 0 \\
* & * & * & * & * & -\varepsilon_4 I & 0 \\
* & * & * & * & * & * & -\varepsilon
\end{bmatrix} < 0 \quad (4)
\]

where \( \Sigma_{11} = PA + A^T P + R, \Sigma_{44} = -C^T PB - B^T PC - (1 - \tilde{h}_d)R, \tilde{E}^T = [E_a^T E_a^T E_b^T E_b^T] \) and \( \varepsilon = \text{diag}[\varepsilon_1 I, \varepsilon_2 I, \varepsilon_3 I, \varepsilon_4 I] \).

**Proof** Choose a legitimate Lyapunov functional \( V(x_t) = V_1 + V_2 \), where

\[
V_1 = D(x_t)^T PD(x_t) \quad (5)
\]

\[
V_2 = \int_{t-h(t)}^{t} x^T(s)Rx(s)ds \quad (6)
\]

where \( D(x_t) = x(t) - Cx(t - h(t)) \).
Observe that $V(x_i)$ satisfies
\[ \lambda_m(P)\|D(x_i)\|^2 \leq V(x_i) \leq [\lambda_M(P) + \tilde{h} \lambda_M(R)]\|x_i\|^2. \quad (7) \]

The time derivative of $V_i$ for $i=1$ and 2, are given, respectively, by
\[ \dot{V}_1 = D^T(x_i)P \dot{D}(x_i) + D^T(x_i)PD(x_i) \]
\[ = 2D^T(x_i)P \dot{D}(x_i) \]
\[ = 2(x^T(t) - x^T(t - h(t))C^T)P(\tilde{A}x(t) + \tilde{B}x(t - h(t))) \]
\[ = 2(x^T(t)P \tilde{A}x(t) + x^T(t)P \tilde{B}x(t - h(t)) - x^T(t - h(t))C^T P \tilde{A}x(t) \]
\[ - x^T(t - h(t))C^T P \tilde{B}x(t - h(t))) \],
\[ \dot{V}_2 = x^T(t)Rx(t) - (1 - \tilde{h}(t))x^T(t - h(t))Rx(t - h(t)) \]
\[ \leq x^T(t)Rx(t) - (1 - \tilde{h}_d(t))x^T(t - h(t))Rx(t - h(t)), \quad (9) \]

where $\tilde{A} = A + \Delta A(t)$ and $B = B + \Delta B(t)$.

From (8) and (9), we have
\[ \dot{V} \leq \chi^T(t) \begin{bmatrix} P \tilde{A} + \tilde{A}^T P + R & P \tilde{B} - \tilde{A}^T PC \\ \ast & -C^T P \tilde{B} - B^T PC - (1 - \tilde{h}_d)R \end{bmatrix} \chi(t), \]
\[ = \chi^T(t) \left[ M + \begin{bmatrix} (PD_aF_a(t)E_a) + (D_aF_a(t)E_a)^T P \\ 0 \end{bmatrix} \begin{bmatrix} PD_bF_b(t)E_b \\ -(D_bF_b(t)E_b)^T PC \end{bmatrix} \right] \chi(t) \]
\[ = \chi^T(t) \left[ M + \tilde{D}_1 F_a(t) \tilde{E}_1 + \tilde{E}_1^T F_a^T(t) \tilde{D}_1^T + \tilde{D}_2 F_a(t) \tilde{E}_1 + \tilde{E}_1^T F_a^T(t) \tilde{D}_2^T \right. \]
\[ \left. + \tilde{D}_1 F_b(t) \tilde{E}_2 + \tilde{E}_2^T F_b^T(t) \tilde{D}_1^T + \tilde{D}_2 F_b(t) \tilde{E}_2 + \tilde{E}_2^T F_b^T(t) \tilde{D}_2^T \right] \chi(t) \quad (10) \]

where $\chi^T(t) = [x^T(t) \quad x^T(t - h(t))]$ and
\[ M = \begin{bmatrix} PA + A^T P + R & PB - A^T PC \\ \ast & -C^T PB - B^T PC - (1 - \tilde{h}_d)R \end{bmatrix}, \]
\[ \tilde{D}_1 = \begin{bmatrix} PD_a \\ 0 \end{bmatrix}, \quad \tilde{D}_2 = \begin{bmatrix} 0 \\ -C^T PD_b \end{bmatrix}, \quad \tilde{E}_1 = [E_a \quad 0], \quad \tilde{E}_2 = [E_b \quad 0]. \]

Using the known fact that
\[ U \Delta V^T + V \Delta U^T \leq \varepsilon UU^T + \varepsilon^{-1}VV^T, \quad \varepsilon > 0 \]
for any matrices $U$, $V$ and $\Delta$ with $\Delta^T \Delta \leq I$, we can eliminate the unknown factor, $F_a(t)$ and $F_b(t)$, in (10). Then, we have

$$
\dot{D}_1 F_a(t) \dot{E}_1 + \dot{E}_1^T F_a^T(t) \dot{D}_1^T \leq \varepsilon_1 \dot{D}_1 \dot{D}_1^T + \varepsilon_1^{-1} \dot{E}_1^T \dot{E}_1 \\
\dot{D}_2 F_a(t) \dot{E}_1 + \dot{E}_1^T F_a^T(t) \dot{D}_2^T \leq \varepsilon_2 \dot{D}_2 \dot{D}_2^T + \varepsilon_2^{-1} \dot{E}_1^T \dot{E}_1 \\
\dot{D}_1 F_b(t) \dot{E}_2 + \dot{E}_2^T F_b^T(t) \dot{D}_1^T \leq \varepsilon_3 \dot{D}_1 \dot{D}_1^T + \varepsilon_3^{-1} \dot{E}_2^T \dot{E}_2 \\
\dot{D}_2 F_b(t) \dot{E}_2 + \dot{E}_2^T F_b^T(t) \dot{D}_2^T \leq \varepsilon_4 \dot{D}_2 \dot{D}_2^T + \varepsilon_4^{-1} \dot{E}_2^T \dot{E}_2
$$

(11)

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4$ are positive scalars to be found later.

Using (11), we obtain a new bound of $\dot{V}$ as

$$
\dot{V} \leq \chi(t)(M + \varepsilon_1 \dot{D}_1 \dot{D}_1^T + \varepsilon_1^{-1} \dot{E}_1^T \dot{E}_1 + \varepsilon_2 \dot{D}_2 \dot{D}_2^T + \varepsilon_2^{-1} \dot{E}_1^T \dot{E}_1 + \varepsilon_3 \dot{D}_1 \dot{D}_1^T \\
+ \varepsilon_2^{-1} \dot{E}_2^T \dot{E}_2 + \varepsilon_4 \dot{D}_2 \dot{D}_2^T + \varepsilon_4^{-1} \dot{E}_2^T \dot{E}_2) \chi(t) \\
\equiv \chi^T(t) \Omega \chi(t).
$$

(12)

Hence, if the matrix $\Omega < 0$, the $\dot{V}$ is negative.

By Lemma 1, the fact that $\Omega < 0$ is equivalent to the following inequality:

$$
\begin{bmatrix}
M + \varepsilon_1 \dot{D}_1 \dot{D}_1^T + \varepsilon_2 \dot{D}_2 \dot{D}_2^T \\
+ \varepsilon_3 \dot{D}_1 \dot{D}_1^T + \varepsilon_4 \dot{D}_2 \dot{D}_2^T \\
\varepsilon_1 I & 0 & 0 & 0 \\
0 & -\varepsilon_1 I & 0 & 0 \\
0 & 0 & -\varepsilon_2 I & 0 \\
0 & 0 & 0 & -\varepsilon_3 I \\
0 & 0 & 0 & -\varepsilon_4 I \\
\end{bmatrix}
< 0
$$

$$
\begin{bmatrix}
PA + A^T P + R \\
+ \varepsilon_1 PD_a D_a^T P + \varepsilon_3 PD_a D_a^T P \\
PB - A^T PC \\
+ \varepsilon_2 C^T PD_b D_b^T P \\
E^T_a & E^T_a & E^T_b & E^T_b \\
-\varepsilon_1 I & 0 & 0 & 0 \\
0 & -\varepsilon_2 I & 0 & 0 \\
0 & 0 & -\varepsilon_3 I & 0 \\
0 & 0 & 0 & -\varepsilon_4 I \\
\end{bmatrix}
> 0
$$

(13)
Also, by Lemma 1, the inequality (13) is equivalent to (4). Hence, if the matrix \( \Sigma < 0 \), it follows that there exists a positive scalar \( \delta \) such that

\[
\dot{V}(x_t) \leq -\delta\|x(t)\|^2 < 0. \tag{14}
\]

Note that \( \rho(C) < 1 \), this implies that the operator \( D(x_t) = x(t) - Cx(t - h(t)) \) is stable [5]. Thus, by Theorem 9.8.1 in [5] with (7) and (14), we conclude that the systems given in (1) are asymptotically stable. This completes the proof. \( \blacksquare \)

**Corollary 1**  
In the case where there is no uncertainty, i.e., \( \Delta A(t) = 0 \) and \( \Delta B(t) = 0 \), then the criterion (4) is simplified as

\[
\begin{bmatrix}
PA + A^T P + R & PB - A^T PC \\
-CT PB - B^T PC - (1 - \tilde{h}_d)R
\end{bmatrix} < 0.
\]

The proof is obvious from the proof of Theorem 1, and omitted.

**Remark 1**  
The inequality given in (4) can be easily solved using Matlab’s LMI Control Toolbox [4]. The LMI Control Toolbox implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [1].

Now, we present a numerical example for application of the proposed delay-independent criterion.

**Example 1**  
Consider the following system:

\[
\dot{x}(t) = \begin{bmatrix}
0.2 & 0 \\
0 & 0.2
\end{bmatrix} \dot{x}(t - h(t)) = \begin{bmatrix}
-2 + \alpha_1 \cos t & 0 \\
0 & -1.5 + \alpha_2 \sin t
\end{bmatrix} x(t) + \begin{bmatrix}
1 + \beta_1 \cos t & 0 \\
-1 & -1 + \beta_2 \sin t
\end{bmatrix} x(t - h(t))
\]

where \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are unknown parameters satisfying

\[
|\alpha_1| \leq 0.5, \quad |\alpha_2| \leq 0.3, \quad |\beta_1| \leq 0.15, \quad |\beta_2| \leq 0.2
\]

and it is assumed that

\[
\dot{h}(t) \leq 0.5.
\]

The above system is the form of (1)–(3) with

\[
A = \begin{bmatrix}
-2 & 0 \\
0 & -1.5
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
-1 & -1
\end{bmatrix}
\]

\[
D_a = E_a = \sqrt{0.5} \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad D_b = E_b = \sqrt{0.15} \begin{bmatrix}
0 \\
0.2
\end{bmatrix}.
\]
By solving the inequality given in (4), we obtain the following solutions of the inequality:

\[ P = \begin{bmatrix} 1.7180 & 0.1630 \\ 0.1630 & 0.4531 \end{bmatrix}, \quad R = \begin{bmatrix} 2.5503 & 0.4140 \\ 0.4140 & 0.8257 \end{bmatrix} \]

\[ \varepsilon_1 = 44.0761, \quad \varepsilon_2 = 42.2419, \quad \varepsilon_3 = 45.8387, \quad \varepsilon_4 = 42.2735. \]

This implies that the above neutral system is robustly stable independent of delay.

3. CONCLUDING REMARKS

This article has developed a new delay-independent criterion for robust stability of a class of uncertain neutral delay-differential systems with time-varying delay using Lyapunov method. The criterion is expressed in terms of a LMI to find the less conservative condition. The solutions of LMI can be easily obtained using various efficient optimization algorithms. A numerical example has been given to illustrate the application of the presented result.

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References

