Guaranteed cost stabilization of neutral differential systems with parametric uncertainty

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Abstract

In this paper, we address the problem of the guaranteed cost stabilization for a class of neutral systems with parametric uncertainties and a given quadratic cost function. The parametric uncertainties are real time-varying norm bounded and state delay is a constant. The problem is to design the state feedback control laws such that the closed-loop system is asymptotically stable and the closed-loop cost function value is not more than a specified upper bound for all admissible uncertainty and delay. Two criteria for the existence of such controllers are derived based on the matrix inequality approach combined with the Lyapunov method. The developed guaranteed cost controllers can be synthesized in terms of the feasible solutions to the certain matrix inequalities.

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1. Introduction

During the three decades, the stability analysis and stabilization problem for uncertain dynamic systems with delay has received much attention. Delay occurs in many dynamic systems such as communication systems, biological systems, chemical systems and electrical networks, and is frequently a source of instability and performance degradation of systems. Especially, in recent years, considerable attention has been focused on the stability analysis of various neutral differential systems [7,10,6]. The theory of neutral delay-differential systems is of both theoretical and practical interest.

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For example, functional differential equations of neutral type are the natural models of fluctuations of voltage and current in problems arising in transmission lines. Also, the neutral systems often appear in the study of automatic control, population dynamics, and vibrating masses attached to an elastic bar. In the literature, numerous criteria for stability of neutral systems have been presented using a number of different analysis techniques by many researchers [9,11,12,8,15–17], while only a few works on controller design for stabilization of the systems has been explored by some researchers [3,19,4,13].

On the other hand, when controlling a real system, it is also desirable to design a control systems which are not only asymptotically stable but also guarantee an adequate level of performance. One design approach to this problem is the so-called guaranteed cost control [2]. The approach has the advantage of providing an upper bound on a given performance index and thus the system performance degradation incurred by uncertainty and delay is guaranteed to be less than this bound. Recently, the design problem of guaranteed cost controller for several class of dynamic systems have been addressed [18,14,20]. Unfortunately, at the knowledge of the author, no paper treats the guaranteed cost control problem for neutral system subjected to uncertainties.

This paper contributes to the development of guaranteed cost stabilization of a class of neutral differential systems with parametric uncertainties. The uncertainty is assumed to be norm bounded and time varying. Using the Lyapunov stability theory combined with matrix inequality technique, we propose a memoryless state feedback controller for the stabilization of the system, which makes the closed-loop system asymptotically stable and guarantees an adequate level of performance. Two stabilization criteria for the existence of the guaranteed cost controller are derived in terms of matrix inequalities, and their solutions provide a parameterized representation of the control. The matrix inequality can be easily solved by various efficient convex optimization algorithms [1]. An example is provided to illustrate the approach developed in this paper.

**Notations:** In the sequel, we denote by $W^T$ and $W^{-1}$ the transpose and the inverse of any square matrix $W$. $\mathbb{R}^n$ denotes $n$-dimensional Euclidean space, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, $I$ denotes identity matrix of appropriate order, and $*$ represents the elements below the main diagonal of a symmetric block matrix. $\| \cdot \|$ denotes Euclidean norm of a given vector. $\text{tr}(\cdot), \lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the trace, minimum and maximum eigenvalue of the matrix $(\cdot)$, respectively. $\text{diag}\{ \cdots \}$ denote the block diagonal matrix. The notation $W > 0$ ($W < 0$) denotes a positive- (negative-) definite matrix $W$.

2. **Problem statement and main result**

In this paper, we consider a class of linear neutral differential system with parametric uncertainty represented by

$$\dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + (A_1 + \Delta A_1(t))x(t-h) + C\dot{x}(t-h) + Bu(t),$$

with the initial condition function

$$x(t_0 + \theta) = \phi(\theta), \quad \forall \theta \in [-h,0],$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A_0, A_1, C \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant system matrices, $u(t) \in \mathbb{R}^m$ is a control vector, $\phi(\cdot)$ is the given continuously differentiable function on
$[-h,0]$, and $\Delta A_0(t)$ and $\Delta A_1(t)$ are unknown real norm-bounded matrix functions which represent time-varying parameter uncertainties. In this paper, it is assumed that the pair $(A_0,B)$ is completely controllable, and the admissible uncertainties are assumed to be of the form

$$\Delta A_0(t) = D_0 F_0(t) E_0, \quad \Delta A_1(t) = D_1 F_1(t) E_1$$

where $D_0, D_1, E_0$ and $E_1$ are real constant matrices with appropriate dimensions, and $F_0(t) \in \mathbb{R}^{k_a \times l_a}$ and $F_1(t) \in \mathbb{R}^{k_b \times l_b}$ are unknown matrices satisfying

$$\|F_0(t)\| \leq 1, \quad \|F_1(t)\| \leq 1.$$

Suppose that system (1) is controlled on the basis of a memoryless state feedback law of the form

$$u(t) = -B^T P x(t),$$

for the purpose of minimizing the following quadratic cost function:

$$J = \int_0^\infty (x^T(t) Q x(t) + u^T(t) S u(t)) \, dt,$$

where $P \in \mathbb{R}^{n \times n} > 0$ to be designed later, and $Q \in \mathbb{R}^{n \times n} > 0$ and $S \in \mathbb{R}^{m \times m} > 0$ are given.

Here, the objective of this article is to develop a procedure to design a state feedback controller $u(t)$ for the system (1) and cost function (5), such that the resulting closed-loop subsystem given by

$$\dot{x}(t) = [A_0 + \Delta A_0(t) - BB^T P] x(t) + [A_1 + \Delta A_1(t)] x(t - h) + C \dot{x}(t - h)$$

is asymptotically stable and the closed-loop value of the cost function (5) satisfies $J \leq J^*$, where $J^*$ is some specified constant.

**Definition 1.** For the uncertain neutral system (1) and cost function (5), if there exist a control law $u^*(t)$ and a positive constant $J^*$ such that for all admissible uncertainty, the closed-loop system (6) is asymptotically stable and the closed-loop value of the cost function (5) satisfies $J \leq J^*$, then $J^*$ is said to be a guaranteed cost and $u^*(t)$ is said to be a guaranteed cost control law of the system (1) and cost function (5).

Throughout the paper, the following well-known facts are needed.

**Fact 1.** Given any real vector $D$ and $E$ with appropriate dimension and any positive scalar $\delta$, the following inequality holds:

$$DE + E^T D^T \leq \delta DD^T + \delta^{-1} E^T E.$$

**Fact 2.** (Schur complements). Given constant symmetric matrices $\Sigma_1, \Sigma_2, \Sigma_3$ where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$
The following theorem, a main result of the paper, gives a delay-independent criterion in terms of matrix inequality, for guaranteed cost stabilization of system (1).

**Theorem 1.** Let \( \tau_0 = \sqrt{x_0(D_0^T D_0)} \) and \( \tau_1 = \sqrt{x_0(D_1^T D_1)} \). For given \( Q > 0 \) and \( S > 0 \), consider system (1) and cost function (5) and suppose that there exist positive scalars \( \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{10}, \beta \), and a positive-definite matrix \( X \) satisfying the following matrix inequality:

\[
\Omega(X, \beta, \varepsilon) = \begin{bmatrix}
\Omega_{1,1} & \Omega_{1,2} & A_1 + XA_0^T A_1 - BB^T A_1 & 0 & C + XA_0^T C - BB^T C \\
* & \Omega_{2,2} & 0 & 0 & 0 \\
* & * & \Omega_{3,3} & \Omega_{3,4} & A_1^T C \\
* & * & * & \Omega_{4,4} & 0 \\
* & * & * & * & \Omega_{5,5}
\end{bmatrix} < 0,
\]

where \( X = P^{-1} \) and

\[
\Omega_{1,1} = XA_0^T + A_0 X - 2BB^T + BB^T BB^T - XA_0^T BB^T - BB^T A_0 X + BS^T
\]
\[
+ \varepsilon_0 D_0 D_0^T + \varepsilon_4 BB^T D_0 D_0^T BB^T,
\]

\[
\Omega_{1,2} = [XA_0^T \beta X Q XQ^T XE_0^T \varepsilon_2 XA_0^T D_0 XE_0^T XA_0^T D_1 XE_0^T XE_0^T XE_0^T \varepsilon_2 XE_0^T XE_0^T \varepsilon_2 XE_0^T D_1 BB^T D_1],
\]

\[
\Omega_{2,2} = \text{diag}\{-I, -\beta I, -Q, -\varepsilon_9 I, -\varepsilon_2 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_5 I, -\varepsilon_7 I, -\varepsilon_7 I, -\varepsilon_8 I, -\varepsilon_8 I\},
\]

\[
\Omega_{3,3} = A_1^T A_1 - \beta I + \varepsilon_1 E_1^T E_1 + \varepsilon_3 E_1^T E_1 + \varepsilon_5 A_1^T D_0 D_0^T A_1 + \varepsilon_6 x_1^2 E_1^T E_1
\]
\[
+ \varepsilon_8 E_1^T E_1 + \varepsilon_9 A_1^T D_1 D_1^T A_1 + x_1^2 E_1^T E_1,
\]

\[
\Omega_{3,4} = [E_1^T E_1^T],
\]

\[
\Omega_{4,4} = \text{diag}\{-\varepsilon_9 I, -\varepsilon_10 I\},
\]

\[
\Omega_{5,5} = C^T C - I + \varepsilon_7 C^T D_0 D_0^T C + \varepsilon_10 C^T D_1 D_1^T C.
\]

Then the system (1) is asymptotically stable by a guaranteed cost control law \( u(t) = -B^T P x(t) \) and the quadratic cost function \( J \) satisfies the bound

\[
J^* = x^T(0) X^{-1} x(0) + \int_{-h}^0 x^T(s) \dot{x}(s) ds + \beta \int_{-h}^0 x^T(s) x(s) ds.
\]

**Proof.** Define a legitimate Lyapunov functional candidate [7]

\[
V = V_1 + V_2 + V_3,
\]
where

\[ V_1 = x^T(t)Px(t), \]  
\[ V_2 = \int_{t-h}^{t} \dot{x}^T(s)\dot{x}(s) \, ds, \]  
\[ V_3 = \beta \int_{t-h}^{t} x^T(s)x(s) \, ds. \]

Taking the time derivative of \( V \) along the solution of (6), we have

\[ \dot{V}_1 = x^T(t)[(A_0^T + \Delta A_0^T)P + P(A_0 + \Delta A_0) - 2PBB^TP]x(t) + 2x^T(t)P(A_1 + \Delta A_1)x(t - h) + 2x^T(t)PC\dot{x}(t - h) \]  
\[ \dot{V}_2 = \dot{x}^T(t)\dot{x}(t) - \dot{x}^T(t - h)\dot{x}(t - h) \]  
\[ = x^T(t)A_0^T A_0 x(t) + x^T(t)PBB^TPx(t) + x^T(t - h)A_1^TA_1 x(t - h) + x^T(t - h)C^T C\dot{x}(t - h) - 2x^T(t)A_0^T BB^TPx(t) + 2x^T(t)A_1^T A_1 x(t - h) + 2x^T(t)A_0^T C\dot{x}(t - h) - 2x^T(t)PBB^TA_1 x(t - h) - 2x^T(t)PBB^TC\dot{x}(t - h) + 2x^T(t - h)A_1^T C\dot{x}(t - h) + 2x^T(t - h)A_0^T \Delta A_0 x(t) + 2x^T(t)\dot{A}_1^T A_1 x(t - h) + x^T(t)\Delta A_0^T \Delta A_0 x(t) - 2x^T(t)A_0^T BB^TPx(t) + 2x^T(t)\dot{A}_0^T A_1 x(t - h) + 2x^T(t)\Delta A_0^T A_1 x(t - h) + 2x^T(t)\Delta A_0^T C\dot{x}(t - h) - 2x^T(t)PBB^T A_1 x(t - h) + 2x^T(t - h)A_1^T A_1 x(t - h) + x^T(t - h) \Delta A_1^T A_1 x(t - h) + 2x^T(t - h) \Delta A_1^T C\dot{x}(t - h) \]  
\[ \dot{V}_3 = \beta x^T(t)x(t) - \beta \dot{x}^T(t - h)x(t - h). \]

Using Fact 1, the some terms on right-hand side of (12) and (13) satisfy the following inequalities:

\[ 2x^T(t)P \Delta A_0 x(t) = 2x^T(t)PD_0 F_0(t) E_0 x(t) \leq \epsilon_0 x^T(t)PD_0 D_0^TPx(t) + \epsilon_0^{-1} x^T(t)E_0^T E_0 x(t), \]
\[ 2x^T(t)P \Delta A_1 x(t) = 2x^T(t)PD_1 F_1(t) E_1 x(t) \leq \epsilon_1^{-1} x^T(t)PD_1 D_1^TPx(t) + \epsilon_1 x^T(t - h)E_1^T E_1 x(t - h), \]
\[ 2x^T(t)A_0^T \Delta A_0 x(t) \leq \epsilon_2 x^T(t)A_0^T D_0 D_0^T A_0 x(t) + \epsilon_2^{-1} x^T(t)E_0^T E_0 x(t), \]
\[ 2x^T(t)A_0^T \Delta A_1 x(t - h) \leq \epsilon_3^{-1} x^T(t)A_0^T D_0 D_1^T A_0 x(t) + \epsilon_3 x^T(t - h)E_1^T E_1 x(t - h), \]
\[ x^T(t)A_0^T \Delta A_0 x(t) = x^T(t)E_0^T F_0^T(t)D_0^T D_0 F_0(t) E_0 x(t) \leq \lambda_M (D_0^T D_0) x^T(t) E_0^T F_0^T(t) F_0(t) E_0 x(t), \]
\[ x^T(t)A_0^T \Delta A_1 x(t - h) \leq \lambda_M (D_0^T D_0) x^T(t) E_0^T F_0^T(t) F_0(t) E_0 x(t), \]
\[ \leq \frac{\lambda}{2} x^T(t) E_0^T E_0 x(t), \]
\[-2x^T(t)\Delta A_0^TBB^TPx(t) \leq \varepsilon_4 x^T(t)PBB^TD_0D_0^TBB^TPx(t) + \varepsilon_4^{-1}x^T(t)E_0^TE_0x(t), \]
\[2x^T(t)\Delta A_0^T A_1 x(t - h) \leq \varepsilon_5^{-1}x^T(t)E_0^TE_0x(t) + \varepsilon_5 x^T(t - h)A_1^TD_0^TA_1 x(t - h), \]
\[2x^T(t)\Delta A_0^T \Delta A_1 x(t - h) \leq \varepsilon_6^{-1}x^T(t) \Delta A_0^T \Delta A_0 x(t) + \varepsilon_6 x^T(t - h) \Delta A_1^T \Delta A_1 x(t - h) \leq \varepsilon_6^{-1}\gamma_0^2x^T(t)E_0^TE_0x(t) + \varepsilon_6 x^T(t - h)E_1^TE_1x(t - h), \]
\[2x^T(t)\Delta A_0^T C\dot{x}(t) \leq \varepsilon_7^{-1}x^T(t)E_0^TE_0x(t) + \varepsilon_7 x^T(t - h)C^TD_0^DC\dot{x}(t - h), \]
\[-2x^T(t)PBB^T \Delta A_1 x(t - h) \leq \varepsilon_8^{-1}x^T(t)PBB^TD_1^TD_1^TBB^TPx(t) + \varepsilon_8 x^T(t - h)E_1^TE_1x(t - h), \]
\[2x^T(t - h)A_1^T \Delta A_1 x(t - h) \leq \varepsilon_9 x^T(t - h)A_1^TD_1^TD_1^TA_1x(t - h) + \varepsilon_9^{-1}x^T(t - h)E_1^TE_1x(t - h), \]
\[x^T(t - h) \Delta A_1^T \Delta A_1 x(t - h) \leq \chi_0^2x^T(t - h)E_1^TE_1x(t - h), \]
\[2x^T(t - h) \Delta A_1^T C\dot{x}(t - h) \leq \varepsilon_{10}^{-1}x^T(t - h)E_1^TE_1x(t - h) + \varepsilon_{10} x^T(t - h)C^TD_1^DC\dot{x}(t - h), \]
\[\frac{d}{dt}V \leq \xi^T(t)M(P, \beta, \varepsilon_i)\xi(t) - x^T(t)(Q + PBSB^TP)x(t), \]  
where $\xi(t) = \begin{bmatrix} x(t) \\ x(t - h) \end{bmatrix}$ and

\[
M(P, \varepsilon_i, \beta) = \begin{bmatrix} M_{11} & \left( PA_1 + A_0^TA_1 \right) & \left( PC + A_0^TC \right) \\ \ast & M_{22} & A_1^TC \\ \ast & \ast & M_{33} \end{bmatrix}
\]

with $M_{11} = A_0^TP + PA_0 - 2PBB^TP + A_0^TA_0 + PBB^TBB^TP - A_0^TBB^TP - PBB^TA_0 + \beta I + Q + PBSB^TP + \varepsilon_0PD_0D_0^TP + \varepsilon_1^{-1}E_0^TE_0 + \varepsilon_2^{-1}PD_1^TD_1^TP + \varepsilon_2 A_0^TD_0D_0^TA_0 + \varepsilon_2^{-1}E_0^TE_0 + \varepsilon_3^{-1}A_0^TD_1^TD_1^TA_0 + \varepsilon_3 E_0^TE_0 + \varepsilon_4 PBB^TD_0D_0^TB^TP + \varepsilon_4^{-1}E_0^TE_0 + \varepsilon_5^{-1}E_1^TE_1 + \varepsilon_6^{-1}C^TE_1E_1 + \varepsilon_6 E_1^TE_1 + \varepsilon_7^{-1}A_0^TD_1^TD_1^TA_1 + \varepsilon_7 E_1^TE_1 + \varepsilon_8^{-1}A_1^TD_1^TD_1^TA_1 + \varepsilon_8 E_1^TE_1 + \varepsilon_9^{-1}A_1^TD_1^TD_1^TA_1 + \varepsilon_9 E_1^TE_1 + \varepsilon_{10}^{-1}E_1^TE_1 + \varepsilon_{10} E_1^TE_1$

and $M_{33} = C^T - I + \varepsilon_7 C^TD_0D_0^TC + \varepsilon_10 C^TD_1^TD_1^TC$.

Therefore, if $M(\cdot) < 0$, there exist a positive scalar $\gamma$ such that

\[\dot{V} < -\gamma\|x(t)\|^2, \quad \gamma = \lambda_m(Q + PBSB^TP), \]

which guarantees the asymptotic stability of the system [7].
Here, pre- and post-multiplying the matrix \( M(\cdot) \) by \( T^T \) and \( T \), where \( T = \text{diag}\{X, I, I\} \), the fact that \( M(\cdot) < 0 \) is equivalent to

\[
\tilde{M}(X, e_i, \beta) = \begin{bmatrix}
M_{11} & \begin{pmatrix} A_1 + XA_0^T A_1 \\ -BB^T A_1 \end{pmatrix} & \begin{pmatrix} C + XA_0^T C \\ -BB^T C \end{pmatrix} \\
* & M_{22} & A_1^T C \\
* & * & M_{33}
\end{bmatrix} < 0,
\tag{18}
\]

where \( \tilde{M}_{11} = XM_{11}X \).

By Fact 2 (Schur complement), it follows that the above inequality (18) is equivalent to the matrix inequality (7).

Also, the matrix inequality (7) implies that

\[
\dot{V} < -x^T(t)(Q + PBSB^T P)x(t) < 0.
\tag{19}
\]

Noting \( Q > 0 \) and \( S > 0 \), this implies that system (6) is asymptotically stable by Lyapunov stability theory. Furthermore, from (19) we have

\[
x^T(t)(Q + PBSB^T P)x(t) < -\dot{V}.
\]

Integrating both sides of the above inequality from 0 to \( T_f \) leads to

\[
\int_0^{T_f} x^T(t)(Q + PBSB^T P)x(t) \, dt < V(0) - V(T_f)
= (x^T(0)Px(0) - x^T(T_f)Px(T_f)) + \left( \int_{-h}^{0} x^T(s)\dot{x}(s) \, ds - \int_{T_f-h}^{T_f} x^T(s)\dot{x}(s) \, ds \right) \\
+ \left( \beta \int_{-h}^{0} x^T(s)x(s) \, ds - \beta \int_{T_f-h}^{T_f} x^T(s)x(s) \, ds \right).
\]

As the closed-loop system (6) is asymptotically stable, when \( T_f \to \infty \),

\[
x^T(T_f)Px(T_f) \to 0, \quad \int_{T_f-h}^{T_f} x^T(s)\dot{x}(s) \, ds \to 0,
\]

\[
\beta \int_{T_f-h}^{T_f} x^T(s)x(s) \, ds \to 0.
\]

Hence, we get

\[
\int_0^{\infty} x^T(t)(Q + PBSB^T P)x(t) \, dt \leq V(0)
= x^T(0)Px(0) + \int_{-h}^{0} x^T(s)\dot{x}(s) \, ds + \beta \int_{-h}^{0} x^T(s)x(s) \, ds \triangleq J^*.
\tag{20}
\]

This completes the proof. \( \square \)
Remark 1. Problem (7) is to determine whether the problem is feasible or not. It is called the feasibility problem. Also, the solutions of the problem can be found by solving generalized eigenvalue problem in \( \omega_0, \omega_1, \ldots, \omega_{10}, X \) and \( \beta \), which is a quasiconvex optimization problem. Note that a locally optimal point of a quasiconvex optimization problem with strictly quasiconvex objective is globally optimal. For details, see Boyd et al. [1]. Various efficient convex optimization algorithms can be used to check whether the matrix inequality (7) is feasible. In this paper, in order to solve the matrix inequality, we utilize Matlab’s LMI Control Toolbox [5], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [1].

Theorem 1 presents a method of designing a state feedback guaranteed cost controller. The following theorem presents a method of selecting a controller minimizing the upper bound of the guaranteed cost (20).

Theorem 2. Consider system (6) with cost function (5). If the following optimization problem

\[
\begin{align*}
\min_{X > 0, z > 0, z > 0, r_1 > 0, r_2 > 0, \beta > 0} & \{z + \text{tr}(\Gamma_1) + \text{tr}(\Gamma_2)\} \\
\text{s.t.} & \begin{align*}
(\text{i}) & \text{ matrix inequality (7),} \\
(\text{ii}) & \begin{bmatrix}
-x & x^T(0) \\
x(0) & -X
\end{bmatrix} < 0, \\
(\text{iii}) & \begin{bmatrix}
-\Gamma_1 & \beta \mathcal{M}^T \\
\beta \mathcal{M} & -\beta I
\end{bmatrix} < 0, \\
(\text{iv}) & \begin{bmatrix}
-\Gamma_2 & \mathcal{N}^T \\
\mathcal{N} & -I
\end{bmatrix} < 0
\end{align*}
\end{align*}
\]

has a positive solution set \((X, \omega, \beta, \alpha, \Gamma_1, \Gamma_2)\), then the control law (4) is an optimal robust guaranteed cost control law which ensures the minimization of the guaranteed cost (20) for the neutral system (6), where \( \int_{-h}^{0} \dot{x}(s)x^T(s)ds = \mathcal{N} \mathcal{A}^T \) and \( \int_{-h}^{0} x(s)x^T(s)ds = \mathcal{M} \mathcal{M}^T \).

Proof. By Theorem 1, (i) in (21) is clear. Also, it follows from the Fact 2 that (ii)–(iv) in (21) are equivalent to \( x^T(0)X^{-1}x(0) < \alpha, \beta \mathcal{M}^T \mathcal{M} < \Gamma_1, \) and \( \mathcal{N}^T \mathcal{N} < \Gamma_2, \) respectively. On the other hand,

\[
\begin{align*}
\beta \int_{-h}^{0} x^T(s)x(s)ds &= \beta \int_{-h}^{0} \text{tr}(x^T(s)x(s))ds = \text{tr}(\beta \mathcal{M}^T \mathcal{M}) \\
&= \text{tr}(\beta \mathcal{M}^T \mathcal{M}) < \text{tr}(\Gamma_1), \\
\int_{-h}^{0} \dot{x}(s)x(s)ds &= \int_{-h}^{0} \text{tr}(\dot{x}^T(s)x(s))ds = \text{tr}(\mathcal{N} \mathcal{A}^T) \\
&= \text{tr}(\mathcal{N} \mathcal{A}^T) < \text{tr}(\Gamma_2).
\end{align*}
\]
Hence, it follows from (20) that

\[ J^* < z + \text{tr}(\Gamma_1) + \text{tr}(\Gamma_2). \]

Thus, the minimization of \( z + \text{tr}(\Gamma_1) + \text{tr}(\Gamma_2) \) implies the minimization of the guaranteed cost for system (6). In the light of Remark 1, this quasiconvex optimization problem guarantees that a global optimum, when it exists, is reachable. □

**Remark 2.** Ma et al. [13] investigated the stabilization problem of a class of neutral systems. However, their method is only applicable to the system with single input and have restrictions on the structure of system matrices \( A_1 \) and \( C \).

3. An example

In order to illustrate the design procedure of the proposed method, we have run a numerical example. Consider the following uncertain system:

\[
\dot{x}(t) = [A_0 + D_0 F_0(t) E_0] x(t) + [A_1 + D_1 F_1(t) E_1] x(t-h) + C \dot{x}(t-h) + Bu(t),
\]

where

\[
A_0 = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & -0.2 \\ -0.2 & 0.4 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix},
\]

\[
D_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad h = 1
\]

\[ \|F_0(t)\| \leq I, \quad \|F_1(t)\| \leq I \]

and the initial condition of the system is as follows:

\[ x(t) = [0.5e^t - 0.5e^{-t}]^T, \quad \text{for} \quad -1 \leq t \leq 0. \]

Actually, when the control input is not forced to the system (22), i.e., \( u(t)=0 \), the system is unstable since the states of the system go to infinity as \( t \to \infty \).

Here, we are about to construct a memoryless state feedback controller of the form (4) for system (22) such that the closed-loop system is asymptotically stable and a corresponding upper bound of the cost function

\[ J = \int_0^\infty (x_1^2(t) + x_2^2(t) + 0.1u^2(t)) \, dt \]

is minimized. That is, \( Q=I \) and \( S=0.1I \).

From the relations (21), \( \int_{-h}^0 x(s)x^T(s) \, ds = \mathcal{N}_1 \mathcal{N}_1^T \) and \( \int_{-h}^0 \dot{x}(s)\dot{x}^T(s) \, ds = \mathcal{N}_2 \mathcal{N}_2^T \), we have

\[
\mathcal{N}_1 = \begin{bmatrix} 0.2380 & -0.2268 \\ -0.2268 & 0.8644 \end{bmatrix}, \quad \mathcal{N}_2 = \begin{bmatrix} 0.2380 & 0.2268 \\ 0.2268 & 0.8644 \end{bmatrix}.
\]
By solving the optimization problem of Theorem 2 using the software LMI toolbox in Matlab [5], it is found that the problem is feasible, and the solutions are

\[
X = \begin{bmatrix}
0.0780 & 0.1315 \\
0.1315 & 0.3258 \\
\end{bmatrix}, \quad \beta = 17.4706, \quad \varepsilon_0 = 0.0625, \quad \varepsilon_1 = 16.1531,
\]

\[
\varepsilon_2 = 5.6139 \times 10^8, \quad \varepsilon_3 = 1.1815, \quad \varepsilon_4 = 4.9023 \times 10^8 \quad \varepsilon_5 = 2.7055, \quad \varepsilon_6 = 3.3419,
\]

\[
\varepsilon_7 = 6.0148, \quad \varepsilon_8 = 3.5105, \quad \varepsilon_9 = 3.4416, \quad \varepsilon_{10} = 5.3050, \quad \alpha = 20.5023,
\]

\[
\Gamma_1 = \begin{bmatrix}
1.8883 & -4.3676 \\
-4.3676 & 13.9526 \\
\end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix}
0.1081 & 0.2500 \\
0.2500 & 0.7986 \\
\end{bmatrix}.
\]

Therefore, the stabilizing optimal guaranteed cost control law, \( u(t) \), for the system (22) is given by

\[
u(t) = -B^T P x(t) = -B^T X^{-1} x(t) = -[12.9382 \quad -7.6782] x(t),
\]

and the corresponding optimal guaranteed cost of the closed-loop system is:

\[J^* = \alpha + \text{tr}(\Gamma_1) + \text{tr}(\Gamma_2) = 37.2498.\]

For computer simulation, we have employed the following uncertainty:

\[
F_0(t) = \cos t, \quad F_1(t) = \sin t.
\]

The simulation results are given in Figs. 1 and 2. In the figures, one can see that the system is indeed well stabilized.
4. Concluding remarks

This paper has considered linear neutral differential system with parametric uncertainty. Using LMIs technique and Lyapunov stability theory, we presented an approach to the guaranteed cost stabilization by memoryless state feedback controller. The parameter of the controller can be obtained through a optimization problem which can be easily solved by various softwares. Finally, a simulation result is illustrated to show that the neutral system is indeed well stabilized irrespective of uncertainty and time delay.

References


