Delay-independent absolute stability for time-delay Lur’è systems with sector and slope restricted nonlinearities

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Received 12 June 2007; received in revised form 18 February 2008; accepted 10 March 2008
Available online 14 March 2008
Communicated by A.R. Bishop

Abstract
This Letter presents an analysis method for a delay-independent absolute stability of time-delay Lur’è systems with sector and slope restrictions. The proposed method is based on the Lyapunov functions with quadratic form of states and nonlinear functions of the systems. Several criteria are derived in terms of linear matrix inequalities (LMIs). A numerical example is illustrated to show the effectiveness of the proposed method.

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PACS: 05.45.-a
Keywords: Lur’è time-delay systems; Sector and slope restrictions; LMIs

1. Introduction

Lur’è systems with sector bound have been extensively studied by many scientists since Lur’è and Postnikov first proposed the concept of absolute stability [1–14]. On the other hand, time-delays are frequently encountered in many fields of science and engineering, including physics, large-scale systems, complex networks, population dynamics, biology, economy and other areas. Therefore, the problem of absolute stability for time-delay Lur’è control systems has been also investigated [12–14]. Recently, Han [15] and Y. He et al. [16] proposed sufficient conditions in terms of linear matrix inequalities (LMIs) which guarantee the robust stability of time-delay Lur’è system in a bounded sector. Their methods are focused on the delay-dependent stability criteria for the sector bounded Lur’è systems without considering slope bounds.

In this Letter, we investigate the absolute stability for time-delay Lur’è systems with not only sector but also slope restrictions. Singh [7], Haddad and Kapila [8] and Park [10] suggested some methods to deal with the slope restricted nonlinear systems by using Popov multiplier or Lur’è–Postnikov-type Lyapunov function. Then several remarkable criteria using slope-bound information are derived in the literature [7–10,17], but the time-delay is not considered in these works.

This Letter is an extension of a recent work [17] and presents a delay-independent criterion for absolute stability of time-delay Lur’è systems with sector and slope-restrictions by using convexity. The method is based on a Lyapunov function with quadratic form of augmented vector which consists of state vector and nonlinear functions of the systems. The nonlinear functions are expressed as convex combinations of sector and slope bounds, so that the equality constraint is derived by using convex properties of the nonlinear function. Then, as a result, new less conservative LMI criteria are derived by using Finsler’s Theorem [11].

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doi:10.1016/j.physleta.2008.03.012
Finally, the effectiveness of the proposed method is shown by two numerical examples.

Through the Letter, $\mathcal{R}^n$ denotes $n$-dimensional Euclidean space, and $\mathcal{R}^{n \times m}$ is the set of all $n \times m$ real matrices. $X < 0$ means that $X$ is a real symmetric negative definite matrix. $I$ denotes the identity matrix with appropriate dimensions. $\|\cdot\|$ refers to Euclidean vector norm or the induced matrix 2-norm.

2. Problem statement and preliminaries

Consider the following time-delay Lur'e system with sector and slope restricted nonlinearities

\[ \dot{x}(t) = (A + \delta A(t)) x(t) + (B + \delta B(t)) \phi(\sigma(t)) + (A_1 + \delta A_1(t)) x(t - \tau), \]

\[ \sigma(t) = C x(t), \quad \phi(\sigma(t)) \triangleq \begin{bmatrix} \phi_1(\sigma_1(t)) \\ \vdots \\ \phi_m(\sigma_m(t)) \end{bmatrix}, \]

where $x(t) \in \mathcal{R}^n$ is the state vector, $\sigma(t) \in \mathcal{R}^m$ is the output vector, $\tau > 0$ is the time-delay in the system, $A(t), A_1 \in \mathcal{R}^{n \times n}, B \in \mathcal{R}^{n \times m}, C \in \mathcal{R}^{m \times n}, A \in \mathcal{R}^{n \times n}$ is asymptotically stable (Hurwitz), and $\phi(\cdot)$ is memoryless time-invariant nonlinearities with sector bound and slope restriction such as

\[ b_i \leq \frac{\phi_i(\sigma)}{\sigma} \leq a_i, \]

\[ b'_i \leq \frac{d\phi_i(\sigma)}{d\sigma} \leq \phi'_i(\sigma) \leq a'_i, \]

in which $b_i$ and $a_i$ are lower/upper sector bound, and $b'_i$ and $a'_i$ are lower/upper slope bounds.

The uncertainties are assumed to be of the following form:

\[ \begin{bmatrix} \delta A(t) & \delta B(t) & \delta A_1(t) \end{bmatrix} = H \{E_1 \quad E_2 \quad E_3\}, \]

where $H, E_1, E_2, E_3$ are known real constant matrices with appropriate dimensions and $F(t)$ is an unknown real time-varying matrix satisfying

\[ \|F(t)\| \leq 1, \quad \forall t. \]

Note that the relationship given in Eq. (4) is fairly standard assumption and widely used in control theory for handling uncertainties.

The normal part of system (1) is given by

\[ \dot{x}(t) = A x(t) + B \phi(\sigma(t)) + A_1 x(t - \tau), \quad \sigma(t) = C x(t). \]

Let us define

\[ \Delta_i(\sigma_i) \triangleq \frac{\phi_i(\sigma_i) - b_i \sigma_i}{(a_i - b_i) \sigma_i} a_i + \frac{a_i q_i - \phi_i(\sigma_i)}{(a_i - b_i) \sigma_i} b_i, \quad \Delta(\sigma) \triangleq \text{diag}\{\Delta_i(\sigma_i), \ldots, \Delta_m(\sigma_m)\}, \]

\[ \tilde{\Delta}(\sigma) \triangleq \text{diag}\{\phi'_i(\sigma_i), \ldots, \phi'_m(\sigma_m)\}. \]

Then the nonlinear function $\phi(\cdot)$ can be expressed as

\[ \phi(\cdot) = \Delta(\sigma) \sigma. \]

For further simplicity, let us also define

\[ \Delta^l \triangleq \text{diag}\{b_1, \ldots, b_m\}, \quad \Delta^u \triangleq \text{diag}\{a_1, \ldots, a_m\}, \]

\[ \tilde{\Delta}^l \triangleq \text{diag}\{b'_1, \ldots, b'_m\}, \quad \tilde{\Delta}^u \triangleq \text{diag}\{a'_1, \ldots, a'_m\}. \]

Then the parameter $\Delta(\sigma)$ belongs to the following set

\[ \Delta := \{\Delta(\sigma) \in \mathcal{R}^{m \times m} \mid \Delta(\sigma) \in \text{Co}\{\Delta^l, \Delta^u\}, \tilde{\Delta}(\sigma) \in \text{Co}\{\tilde{\Delta}^l, \tilde{\Delta}^u\}\}, \]

where Co denotes the convex hull.

Define $\nabla := \text{diag}\{\Delta(\sigma), \tilde{\Delta}(\sigma)\}$ in the following set

\[ \Phi := \{\text{diag}(\Delta, \tilde{\Delta}) \mid \Delta \in \text{Co}\{\Delta^l, \Delta^u\}, \tilde{\Delta} \in \text{Co}\{\tilde{\Delta}^l, \tilde{\Delta}^u\}\}. \]

The following Finsler’s Lemma will be used for deriving an LMI condition of a robust stability of the system (1).
Lemma 1. (See [11].) Let matrices $Q = Q^T$, $F$, and a compact subset of real matrices $H$ be given. The following statements are equivalent:

- for each $H \in H$
  \[
  \xi^T Q \xi < 0 \quad \forall \xi \neq 0 \quad \text{s.t.} \quad HF \xi = 0, \tag{12}
  \]
- there exists $\Theta = \Theta^T$ s.t.
  \[
  Q + F^T \Theta F < 0, \quad N_H^T \Theta N_H \geq 0, \quad \forall H \in H, \tag{13}
  \]
where $N_H$ is a matrix belong to a null space of $H$.

3. Main results

In this section, we derive an LMI condition for absolute stability of time-delay Lur'e systems with sector and slope restrictions by using the convexity of the nonlinear functions.

Let us define the following matrices for stability analysis of the normal system (6) in the following theorem:

\[
\begin{align*}
\Sigma_1 &= \begin{bmatrix}
Q + XA + A^T X & XB + A_1^T Y & XA_1 & Y \\
* & Y^T B + B^T Y & Y^T A_1 & Z \\
* & * & * & -Q \\
* & * & * & 0 \\
\end{bmatrix}, \\
\Sigma_2 &= \begin{bmatrix}
C & 0 & 0 & 0 \\
CA & CB & CA_1 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & 0 & I \\
\end{bmatrix}, \\
P &= \begin{bmatrix}
X & Y \\
Y^T & Z \\
\end{bmatrix}.
\end{align*}
\tag{14}
\]

Theorem 1. Consider the normal system (6). Then, the system (6) is stable for all $\Delta \in \Delta$ if there exist positive symmetric matrices $P, Q$ and $\Theta$ subject to

\[
\begin{align*}
\Sigma_1 + \Sigma_2 \Theta \Sigma_2 &< 0, \\
\begin{bmatrix} I \\ \nabla \end{bmatrix}^T \begin{bmatrix} I \\ \nabla \end{bmatrix} &\geq 0, \quad \forall \nabla \in \Phi. \tag{15}
\end{align*}
\]

Proof. Let a candidate of the Lyapunov function as follows:

\[
V(x(t)) \triangleq x_a^T (t) P x_a(t) + \int_{t-\tau}^{t} x^T (s) Q x(s) \, ds, \tag{17}
\]

where

\[
x_a(t) = \begin{bmatrix} x^T (t) & \phi^T (\sigma (t)) \end{bmatrix}^T.
\]

Then the derivative of the Lyapunov function (17) can be written

\[
\dot{V}(x(t)) = \Phi(t)^T \begin{bmatrix} I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & 0 \\
\end{bmatrix} P \begin{bmatrix} A & B & A_1 \\
0 & 0 & 0 \\
0 & 0 & I \\
\end{bmatrix} + \bar{Q} \Phi(t), \tag{18}
\]

where

\[
\Phi(t) = \begin{bmatrix} x(t) \\ \phi (\sigma (t)) \\
x(t-\tau) \\ \phi (\sigma (t)) \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} Q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -Q & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}.
\tag{19}
\]

Using convex properties of the nonlinear function, we obtain the following constraint:

\[
\begin{bmatrix} \Delta C \\
\Delta CA \\
\Delta CB \\
\Delta CA_1 \\
\end{bmatrix} \Phi(t) = 0.
\tag{20}
\]

The Lyapunov inequality (18) is rewritten by

\[
\Phi(t)^T \Sigma_1 \Phi(t) < 0, \tag{21}
\]
such that
\[
\begin{bmatrix} \nabla - I \end{bmatrix} \Sigma_2 \Phi(t) = 0 \tag{22}
\]
for all $\nabla \in \Phi$.

By Finsler's Theorem [11], the conditions (21) and (22) is satisfied, if there exist a symmetric matrix $\Theta$ satisfying (15) and (16). This completes the proof. □

In the following, for the system (1) with time-varying structured uncertainties, Theorem 1 is extended to the system with uncertainties by using well-known S-procedure [4].

The system (1) is rewritten as
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\phi(\sigma(t)) + A_1 x(t - \tau) + H p(t), \\
q(t) &= E_1 x(t) + E_2 \phi(\sigma(t)) + E_3 x(t - \tau), \\
\sigma(t) &= C x(t). 
\end{align*}
\tag{23}
\]

Define the following matrices:
\[
\Sigma_3 = \begin{bmatrix}
Q + XA + A^T X & XB + A^T_1 Y & X A_1 & XH & Y \\
* & Y^T B + B^T Y & Y^T A_1 & Y^T H & Z \\
* & * & -Q & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & * & * & 0
\end{bmatrix},
\]
\[
\Sigma_4 = \begin{bmatrix}
C & 0 & 0 & 0 & 0 \\
CA & CB & CA_1 & CH & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I
\end{bmatrix}, \quad \bar{E} = [E_1 \ E_2 \ E_3 \ 0 \ 0].
\tag{24}
\]

**Theorem 2.** The system (1) is absolutely stable, if there exist $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$ and $\Theta_2 = \Theta_2^T$ such that the following LMIs hold:
\[
\begin{bmatrix}
\Sigma_3 + \Sigma_4^T \Theta_2 \Sigma_4 & \bar{E}^T R \\
R \bar{E} & -R
\end{bmatrix} < 0, \
\begin{bmatrix}
I \\
\nabla
\end{bmatrix}' \Theta_2 \begin{bmatrix}
I \\
\nabla
\end{bmatrix} \geq 0, \quad \forall \nabla \in \Phi.
\tag{25, 26}
\]

**Proof.** The derivative of the Lyapunov function (17) is written
\[
\dot{V}(x(t)) = \xi^T(t) \begin{bmatrix}
I \\
0 \\
0 \\
0 \\
0
\end{bmatrix} P \begin{bmatrix}
A & B & A_1 & H & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} + \tilde{Q}' \xi(t),
\tag{27}
\]
where
\[
\xi(t) = \begin{bmatrix}
x(t) \\
\phi(t) \\
x(t - \tau) \\
p(t) \\
\phi(t)
\end{bmatrix}, \quad \tilde{Q}' = \begin{bmatrix}
Q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -Q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\tag{28}
\]

Also, the following equality constraint is derived
\[
\begin{bmatrix}
\Delta C & -I \\
\Delta CA & \Delta C B & \Delta CA_1 & \Delta C H & -I
\end{bmatrix} \xi(t) = 0, \quad \begin{bmatrix} \nabla - I \end{bmatrix} \Sigma_4 \xi(t) = 0. 
\tag{29}
\]

Then, the Lyapunov inequality (27) is rewritten by
\[
\xi(t)^T \Sigma_3 \xi(t) < 0,
\tag{30}
\]
such that
\[
\begin{bmatrix} \nabla - I \end{bmatrix} \Sigma_4 \xi(t) = 0
\tag{31}
\]

\[
\begin{bmatrix}
\Delta C & -I \\
\Delta CA & \Delta C B & \Delta CA_1 & \Delta C H & -I
\end{bmatrix} \xi(t) = 0, \quad \begin{bmatrix} \nabla - I \end{bmatrix} \Sigma_4 \xi(t) = 0.
\tag{29}
\]

Then, the Lyapunov inequality (27) is rewritten by
\[
\xi(t)^T \Sigma_3 \xi(t) < 0,
\tag{30}
\]
such that
\[
\begin{bmatrix} \nabla - I \end{bmatrix} \Sigma_4 \xi(t) = 0
\tag{31}
\]
for all $\nabla \in \Phi$. 

If there exist a symmetric matrix $\Theta_2$ satisfying
\[
\Sigma_3 + \Sigma_4^T \Theta_2 \Sigma_4 < 0,
\]
\[
\begin{bmatrix}
I \\
\nabla
\end{bmatrix} \Theta_2 \begin{bmatrix}
I \\
\nabla
\end{bmatrix} \geq 0, \quad \forall \nabla \in \Phi,
\]
then the conditions (30) and (31) are satisfied by the Finsler's Lemma [11].

For any positive symmetric matrix $R_1$, the following inequality is satisfied
\[
p^T(t)R_1 p(t) = q^T(t)F^T(t)R_1 F(t)q(t) \leq q^T(t)R_1 q(t) = (E_1 x(t) + E_2 \phi(\sigma(t)) + E_3 x(t - \tau))^T R_1 (E_1 x(t) + E_2 \phi(\sigma(t)) + E_3 x(t - \tau)) = \zeta^T(t) \tilde{E}^T R_1 \tilde{E} \zeta(t).
\]

By S-procedure [4], if there exist a diagonal matrix $\Lambda > 0$ satisfying
\[
\Sigma_3^T + \Sigma_4 \Theta_2 \Sigma_4 + \Lambda \tilde{E}^T R_1 \tilde{E} < 0,
\]
where
\[
\Sigma_3^T = \begin{bmatrix}
Q + XA + AT X & XB + AT^2 Y & XA_1 & XH Y \\
* & Y^T B + B^T Y & Y^T A_1 & Y^T H Z \\
* & * & -Q & 0 \\
* & * & * & -\Lambda R_1 \\
* & * & * & 0
\end{bmatrix},
\]
the inequalities (32) and (34) are satisfied.

By Shur’s complement [4], the inequality (35) is written as
\[
\begin{bmatrix}
\Sigma_3^T + \Sigma_4 \Theta_2 \Sigma_4 & \tilde{E}^T \\
\tilde{E} & -(\Lambda R_1)^{-1}
\end{bmatrix} < 0.
\]
Define
\[
R = \Lambda R_1,
\]
and multiply by $\text{diag}(I, R)$ at the left and right side in Eq. (37), then inequality (25) is obtained. This completes our proof.

4. Numerical examples

In order to show the effectiveness of the technique proposed in this Letter, we revisit the example in He et al. [16] and compare our delay-independent criterion with existing delay-independent criteria. Also, for the deterministic case with the delay-free, our approach is compared with Criterion 6 of a recent work [18].

**Example 1.** Consider the system (1) with the following matrices
\[
A = \begin{bmatrix}
-1 & 0 \\
1 & -2
\end{bmatrix}, \quad B = \begin{bmatrix}
-0.5 & -0.1 \\
0.1 & -0.5
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}, \quad C = I,
\]
\[
H = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad E_1 = \begin{bmatrix}
0.2 & 0 \\
0 & 0.2
\end{bmatrix}, \quad E_2 = \begin{bmatrix}
0.05 & 0 \\
0 & 0.05
\end{bmatrix}, \quad E_3 = \begin{bmatrix}
0.05 & 0 \cr 0 & 0.05
\end{bmatrix}.
\]

Set the slope bound as $b_1 = 0$, $b_2 = 0$, $a_1' = 0$, $a_2' = a_2$. By keeping the sector bound $b_1 = 0$, $b_2 = 0$, $a_1 = 0.3$ and varying the upper sector bound $a_2$, the maximum feasible solution $a_2$ is found. Table 1 shows the comparison of our method with recent other delay-independent absolute stability conditions. Since Han’s Corollary 9 [15] is delay-dependent condition, we modify the condition to delay-independent condition by setting the parameter $R = 0$. Then, our proposed method is compared with the Han’s Corollary 9 with $R = 0$ [15] and He’s Theorem 5 [16].

One can see from Table 1 that our result is less conservative than existing ones. In the Lyapunov function given in Eq. (17), if we set $Y = 0$ and $Z = 0$, the stability condition becomes the delay-independent circle criterion [15] and if we set just $Y = 0$, the condition becomes the delay-independent Popov criterion [16].
Table 1
Comparison of sector bound $a_2$

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector bound $a_2$</td>
<td>0.23</td>
<td>12.3</td>
<td>28.8</td>
</tr>
</tbody>
</table>

Table 2
Comparison of sector bound $a_1$ with fixed $a_2$ and $a'_1, a'_2 = \infty$

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Criterion 6 of Yang [18]</th>
<th>Theorem 1 with delay free</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_2 = 0.5$</td>
<td>4.7709</td>
<td>4.7709</td>
</tr>
<tr>
<td>$a_2 = 2.5$</td>
<td>1.5363</td>
<td>3.1745</td>
</tr>
</tbody>
</table>

Example 2. For the comparison with delay free deterministic case, consider the example given in [18]

$$A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-3.6364 & 0.6568 & -0.3092 & -1.8480 & -2.4480 & -2.0649 \\
0.5 & -1 & 0.5 & 0 & 0 & 0 \\
0 & 0.5 & -0.5 & 0 & 0 & 0
\end{bmatrix}, \quad (40)$$

$$B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
-1 & 1 \\
0 & -1
\end{bmatrix}, \quad C = \begin{bmatrix}
-1 & 0 \\
1 & -1 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \quad (41)$$

By keeping the sector bound $a_2$, the maximum bound $a_1$ is found. In order to fairly compare, we modified Theorem 1 with delay free and set the slope bound infinity. The simulation results in Table 2 show that the proposed method is less conservative than that of [18].

5. Conclusions

In this Letter, a new analysis method for absolute stability is presented for time-delay Lur’e systems with sector and slope nonlinearities. Based on the Lyapunov method, novel delay-independent criteria are derived in terms of LMIs, which can be easily solved by various convex optimization algorithms. In this Letter, to derive a less conservative condition, a method using slope bound information is proposed by quadratic Lyapunov function which has augmented vector of states and nonlinear functions. Not only normal systems but also the systems with time-varying structured uncertainties are considered in this Letter. Numerical examples demonstrated that the proposed method is less conservative than other existing delay-independent criteria.

References