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Dynamic output guaranteed cost controller for neutral systems with input delay

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Abstract

In this paper, we consider a design problem of dynamic output guaranteed cost controller (GCC) of a class of neutral systems with input delay. A quadratic cost function is considered as a performance measure for the closed-loop system. Based on the Lyapunov second method, two stability criteria for existence of the controller are derived in terms of matrix inequalities. The solutions of the matrix inequalities can be easily obtained using existing efficient convex optimization techniques. A numerical example is given to illustrate the proposed design method.
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1. Introduction

The guaranteed cost control of dynamic systems was first introduced by Chang and Peng [1] and studied extensively by many researchers. The objective of the control problem is to design a controller to asymptotically stabilize the dynamic system and guarantee an adequate level of performance. Recently, Petersen and McFarlane [2] presented a Riccati equation approach for designing quadratic GCC of dynamic systems. Using linear matrix inequality approach, Yu and Chu [3] developed a new design method for designing GCC of dynamic systems with delay in state. Park [4,5] extended the linear matrix inequality approach to neutral delay-differential systems. In certain practical situations, there is a strong need to construct dynamic output feedback controller instead of static controller in order to obtain better performance and dynamical behavior of state response. However, all the controllers developed by Refs. [2,3,4,5] were static feedback controllers. Thus, more recently, the dynamic output GCC problem for dynamic systems have been investigated by some researchers [6–8]. On the other hand, in real control systems, input delays are often encountered because of transmission of the measurement information. The existence of the delay may be a source of instability and poor performance [9]. However, to the best of our knowledge, it seems that there is no previous results on dynamic output GCC for neutral delay-differential systems with input delay.

This paper is concerned with the design problem of GCC of a class of neutral delay systems with input delay. We provide the convex optimization problem for existence of the controller, which renders the robust stability of the closed-loop system and guarantee an adequate level of performance. Since the proposed optimization problem ensures that a

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global optimum is reachable when it exists, all design variables and the upper bound of guaranteed cost can be obtained at the same time. Utilizing the variables, we can easily find a stabilizing dynamic output GCC.

Notation: Through the paper, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, I denotes identity matrix of appropriate order, and $\star$ represents the elements below the main diagonal of a symmetric block matrix. $\rho(\cdot)$ and $\mathrm{Tr}(\cdot)$ denote the spectral radius and trace of given matrix, respectively. $\mathrm{diag}\{\cdot\}$ denotes the block diagonal matrix. The notation $W>0$ ($\geqslant, <, \leqslant 0$) denotes a symmetric positive definite (positive semidefinite, negative, negative semidefinite) matrix W.

2. Problem statement and controller design

Consider a class of neutral delay-differential system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1\dot{x}(t-h) + A_2\int_{t-h}^{t} x(s)\,\mathrm{d}s + B_0u(t) + B_1u(t-h), \\ y(t) = Cx(t) \end{cases} \tag{1}$$

with the initial condition function

$$x(t_0+\theta) = \phi(\theta) \quad \forall\ \theta\in[-h,0], \tag{2}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $A, A_1, A_2 \in \mathbb{R}^{n\times n}$, $B_0, B_1\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{l\times n}$ are constant system matrices, $u(t)\in\mathbb{R}^m$ is the control input, $y(t)\in\mathbb{R}^l$ is the measured output, h is a positive constant time delay, and $\phi(\cdot)\in\mathscr{C}_0$ is the initial vector, where $\mathscr{C}_0$ is a set of all continuous differentiable function on $[-h,0]$ to $\mathbb{R}^n$.

Associated with the system (1) is the following quadratic cost function

$$J = \int_0^\infty (x^{\mathrm{T}}(t)\Gamma_1x(t) + u^{\mathrm{T}}(t)\Gamma_2u(t))\,\mathrm{d}t, \tag{3}$$

where $\Gamma_1\in\mathbb{R}^{n\times n}$ and $\Gamma_2\in\mathbb{R}^{m\times m}$ are given positive-definite matrices.

In this article, we propose the following dynamic output feedback controllers in order to stabilize system (1):

$$\begin{cases} \dot{\xi}(t) = A_c\xi(t) + B_cy(t), \\ u(t) = C_c\xi(t), \quad \xi(0)=0, \end{cases} \tag{4}$$

where $\xi(t)\in\mathbb{R}^n$ is the controller state vector, and A_c, B_c, and C_c are gain matrices with appropriate dimensions to be determined later.

Applying this controller (4) to system (1) results in the closed-loop system

$$\dot{z}(t) = \overline{A}_0z(t) + \overline{B}z(t-h) + \overline{A}_1\dot{z}(t-h) + \overline{A}_2\int_{t-h}^{t} z(s)\,\mathrm{d}s, \tag{5}$$

where

$$z(t) = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \quad \overline{A}_0 = \begin{bmatrix} A & B_0C_c \\ B_cC & A_c \end{bmatrix},$$

$$\overline{B} = \begin{bmatrix} 0 & B_1C_c \\ 0 & 0 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{A}_2 = \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

The corresponding closed-loop cost function is

$$J = \int_0^\infty z^{\mathrm{T}}(t)\begin{bmatrix} \Gamma_1 & 0 \\ 0 & C_c^{\mathrm{T}}\Gamma_2C_c \end{bmatrix}z(t)\,\mathrm{d}t \equiv \int_0^\infty z^{\mathrm{T}}(t)\overline{\Gamma}z(t)\,\mathrm{d}t. \tag{6}$$

Before stating the main results, the following definition, fact and lemma are needed.

Definition 1. For neutral delay system (1) and cost function (3), if there exist a control law $u^*(t)$ and a positive scalar J^* such that the closed-loop system (5) is asymptotically stable and the closed-loop value of the cost function (3) satisfies $J \leqslant J^*$, then J^* is said to be a guaranteed cost and $u^*(t)$ is said to be a GCC of the system (1) and the cost function (3).

Fact 2 (Schur complement). Given constant symmetric matrices Σ_1, Σ_2, Σ_3 where $\Sigma_1 = \Sigma_1^{\mathrm{T}}$ and $0 < \Sigma_2 = \Sigma_2^{\mathrm{T}}$, then $\Sigma_1 + \Sigma_3^{\mathrm{T}}\Sigma_2^{-1}\Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^{\mathrm{T}} \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^{\mathrm{T}} & \Sigma_1 \end{bmatrix} < 0.$$

Lemma 3 [11]. *For any constant symmetric positive-definite matrix Θ, a positive scalar σ, and the vector function $\omega : [0,\sigma] \to \mathbb{R}^m$ such that the integrations in the following are well defined, then*

$$\sigma \int_0^\sigma \omega^{\mathrm{T}}(s)\Theta\omega(s)\,\mathrm{d}s \geqslant \left(\int_0^\sigma \omega(s)\,\mathrm{d}s\right)^{\mathrm{T}} \Theta \left(\int_0^\sigma \omega(s)\,\mathrm{d}s\right).$$

To obtain the main result of the paper, let us rewrite system (5) in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[z(t) + \int_{t-h}^{t} \overline{B}z(s)\,\mathrm{d}s - \overline{A}_1 z(t-h)\right] = \overline{A}z(t) + \overline{A}_2 \int_{t-h}^{t} z(s)\,\mathrm{d}s, \quad t \geqslant 0, \tag{7}$$

where $\overline{A} = \overline{A}_0 + \overline{B}$.

Define a new operator $\mathscr{D}(z_t) : \mathscr{C}_0 \to \mathbb{R}^n$ as

$$\mathscr{D}(z_t) = z(t) + \int_{t-h}^{t} \overline{B}z(s)\,\mathrm{d}s - \overline{A}_1 z(t-h). \tag{8}$$

Remark 4. The well-known criterion [9] for stability of the operator $\mathscr{D}(z_t)$ given in (8) is

$$\rho(h|\overline{B}| + |\overline{A}_1|) < 1. \tag{9}$$

Now, we establish a criterion in terms of matrix inequalities, for dynamic output GCC of the neutral delay-differential system (1) using the Lyapunov stability theory.

Theorem 5. *For given scalar $h > 0$, suppose that $\rho(|A_1|) < 1$. Then, there exists the dynamic output GCC (4) for the system (1) if there exist a positive scalars ϵ, η and δ, positive-definite matrices S, Y, X, and matrices $\widehat{A}, \widehat{B}, \widehat{C}$ satisfying the following matrix inequalities:*

$$\begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & -\widehat{A}^{\mathrm{T}} & -h\cdot\Omega_3 & h\widehat{A}^{\mathrm{T}} & hY & hI & \Omega_6 \\ \star & \Omega_4 & -A^{\mathrm{T}} & \Omega_5 & hA^{\mathrm{T}} & -h\cdot\Omega_5 & hI & hS & 0 \\ \star & \star & -hX & -\epsilon hY & 0 & 0 & -hY & -hI & 0 \\ \star & \star & \star & -\epsilon hI & 0 & 0 & -hI & -hS & 0 \\ \star & \star & \star & \star & -X & -\delta Y & h^2Y & h^2I & 0 \\ \star & \star & \star & \star & \star & -\delta I & h^2I & h^2S & 0 \\ \star & \star & \star & \star & \star & \star & -X & -\eta Y & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\eta I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \Omega_7 \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} Y & I \\ I & S \end{bmatrix} > 0, \tag{11}$$

where

$$\begin{aligned}
B &= B_0 + B_1,\\
\Omega_1 &= AY + YA^{\mathrm{T}} + B\widehat{C} + \widehat{C}^{\mathrm{T}}B^{\mathrm{T}},\\
\Omega_2 &= \widehat{A}^{\mathrm{T}} + A + \epsilon hYA_1^{\mathrm{T}}A_1 + \eta YA_2^{\mathrm{T}}A_2 + Y\Gamma_1,\\
\Omega_3 &= -(YA^{\mathrm{T}} + \widehat{C}^{\mathrm{T}}B^{\mathrm{T}}),\\
\Omega_4 &= SA + A^{\mathrm{T}}S + \widehat{B}C + C^{\mathrm{T}}\widehat{B}^{\mathrm{T}} + \epsilon hA_1^{\mathrm{T}}A_1 + \eta A_2^{\mathrm{T}}A_2 + \Gamma_1,\\
\Omega_5 &= -(A^{\mathrm{T}}S + C^{\mathrm{T}}\widehat{B}^{\mathrm{T}}),\\
\Omega_6 &= \begin{bmatrix}\epsilon hYA_1^{\mathrm{T}} & \delta\widehat{C}^{\mathrm{T}}B_1^{\mathrm{T}} & \eta YA_2^{\mathrm{T}} & Y\Gamma_1 & \widehat{C}^{\mathrm{T}}\Gamma_2\end{bmatrix},\\
\Omega_7 &= \mathrm{diag}\{-\epsilon hI, -\delta I, -\eta I, -\Gamma_1, -\Gamma_2\}.
\end{aligned} \tag{12}$$

Also, the upper bound of cost function (3) is

$$J \leqslant (x(0) - A_1x(-h))^{\mathrm{T}}S(x(0) - A_1x(-h)) + \epsilon h\int_{-h}^{0} x^{\mathrm{T}}(s)A_1^{\mathrm{T}}A_1x(s)\,\mathrm{d}s + \frac{\eta}{h}\int_{-h}^{0}(s+h)x^{\mathrm{T}}(s)A_2^{\mathrm{T}}A_2x(s)\,\mathrm{d}s. \tag{13}$$

Proof. Consider the following legitimate Lyapunov functional candidate:

$$V = \mathscr{D}^{\mathrm{T}}(z_t)P\mathscr{D}(z_t) + h\int_{t-h}^{t} z^{\mathrm{T}}(s)\overline{A}_1^{\mathrm{T}}W_1\overline{A}_1z(s)\,\mathrm{d}s + \frac{1}{h}\int_{t-h}^{t}(s-t+h)z^{\mathrm{T}}(s)(\overline{A}_2^{\mathrm{T}}W_2\overline{A}_2 + \overline{B}^{\mathrm{T}}W_3\overline{B})z(s)\,\mathrm{d}s, \tag{14}$$

where $P > 0$, $W_1 > 0$, $W_2 > 0$ and $W_3 > 0$.

Taking the time derivative of V along the solution of (8), we have

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} = {} & 2\left(\overline{A}z(t) + \int_{t-h}^{t}\overline{A}_2z(s)\,\mathrm{d}s\right)^{\mathrm{T}}P\left(z(t) + \int_{t-h}^{t}\overline{B}z(s)\,\mathrm{d}s - \overline{A}_1z(t-h)\right) + hz^{\mathrm{T}}(t)\overline{A}_1^{\mathrm{T}}W_1\overline{A}_1z(t)\\
& - hz^{\mathrm{T}}(t-h)\overline{A}_1^{\mathrm{T}}W_1\overline{A}_1z(t-h) + z^{\mathrm{T}}(t)(\overline{B}^{\mathrm{T}}PW_3\overline{B} + \overline{A}_2^{\mathrm{T}}W_2\overline{A}_2)z(t) - \frac{1}{h}\int_{t-h}^{t}z^{\mathrm{T}}(s)(\overline{A}_2^{\mathrm{T}}W_2\overline{A}_2 + \overline{B}^{\mathrm{T}}W_3\overline{B})z(s)\,\mathrm{d}s.
\end{aligned} \tag{15}$$

Applying Lemma 3 to the term $-\int_{t-h}^{t}z^{\mathrm{T}}(s)(\overline{B}^{\mathrm{T}}W_3\overline{B} + \overline{A}_2^{\mathrm{T}}W_2\overline{A}_2)z(s)\,\mathrm{d}s$ of right-hand side of (15) gives the following:

$$\begin{aligned}
-\int_{t-h}^{t}z^{\mathrm{T}}(s)\overline{A}_2^{\mathrm{T}}W_2\overline{A}_2z(s)\,\mathrm{d}s &\leqslant -\left(\frac{1}{h}\int_{t-h}^{t}\overline{A}_2z(s)\,\mathrm{d}s\right)^{\mathrm{T}}(hW_2)\left(\frac{1}{h}\int_{t-h}^{t}\overline{A}_2z(s)\,\mathrm{d}s\right),\\
-\int_{t-h}^{t}z^{\mathrm{T}}(s)\overline{B}^{\mathrm{T}}W_3\overline{B}z(s)\,\mathrm{d}s &\leqslant -\left(\frac{1}{h}\int_{t-h}^{t}\overline{B}z(s)\,\mathrm{d}s\right)^{\mathrm{T}}(hW_3)\left(\frac{1}{h}\int_{t-h}^{t}\overline{B}z(s)\,\mathrm{d}s\right).
\end{aligned} \tag{16}$$

Substituting (16) into (15) gives that

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant {} & z^{\mathrm{T}}(t)\left(P\overline{A} + \overline{A}^{\mathrm{T}}P + h\overline{A}_1^{\mathrm{T}}W_1\overline{A}_1 + \overline{A}_2^{\mathrm{T}}W_2\overline{A}_2 + \overline{B}^{\mathrm{T}}W_3\overline{B}\right)z(t) + 2z^{\mathrm{T}}(t)\overline{A}^{\mathrm{T}}P\int_{t-h}^{t}\overline{B}z(s)\,\mathrm{d}s\\
& + 2\left(\overline{A}_2\int_{t-h}^{t}z(s)\,\mathrm{d}s\right)^{\mathrm{T}}Pz(t) + 2\left(\overline{A}_2\int_{t-h}^{t}z(s)\,\mathrm{d}s\right)^{\mathrm{T}}P\int_{t-h}^{t}\overline{B}z(s)\,\mathrm{d}s - 2\left(\overline{A}_2\int_{t-h}^{t}z(s)\,\mathrm{d}s\right)^{\mathrm{T}}P\overline{A}_1z(t-h)\\
& - 2z^{\mathrm{T}}(t)\overline{A}^{\mathrm{T}}P\overline{A}_1z(t-h) - \left(\frac{1}{h}\int_{t-h}^{t}\overline{A}_2z(s)\,\mathrm{d}s\right)^{\mathrm{T}}W_2\left(\frac{1}{h}\int_{t-h}^{t}\overline{A}_2z(s)\,\mathrm{d}s\right) - hz^{\mathrm{T}}(t-h)\overline{A}_1^{\mathrm{T}}W_1\overline{A}_1z(t-h)\\
& - \left(\frac{1}{h}\int_{t-h}^{t}\overline{B}z(s)\,\mathrm{d}s\right)^{\mathrm{T}}W_3\left(\frac{1}{h}\int_{t-h}^{t}\overline{B}z(s)\,\mathrm{d}s\right),\\
\equiv {} & \mathscr{Z}^{\mathrm{T}}(t)\Sigma\mathscr{Z}(t), \quad = \mathscr{Z}^{\mathrm{T}}(t)\left(\Sigma + \mathrm{diag}\{\overline{\Gamma}, 0, 0, 0\}\right)\mathscr{Z}(t) - z^{\mathrm{T}}(t)\overline{\Gamma}z(t),
\end{aligned} \tag{17}$$

where

$$\mathscr{Z}(t) = \begin{bmatrix} z(t) \\ \overline{A}_1 z(t-h) \\ \frac{1}{h}\int_{t-h}^{t} \overline{B} z(s)\,\mathrm{d}s \\ \frac{1}{h}\int_{t-h}^{t} \overline{A}_2 z(s)\,\mathrm{d}s \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} 2P\overline{A} + h\overline{A}_1 W_1 \overline{A}_1 + \overline{A}_2^{\mathrm{T}} W_2 \overline{A}_2 + \overline{B}^{\mathrm{T}} W_3 \overline{B} & -\overline{A}^{\mathrm{T}} P & h\overline{A}^{\mathrm{T}} P & hP \\ \star & -hW_1 & 0 & -hP \\ \star & \star & -W_3 & h^2 P \\ \star & \star & \star & -W_2 \end{bmatrix}.$$

Thus, if the inequality $\overline{\Sigma} = \Sigma + \mathrm{diag}\{\overline{\Gamma}, 0, 0, 0\} < 0$ holds, there exists a positive scalar γ such that

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant -\gamma \|z(t)\|^2 < 0. \tag{18}$$

By Remark 4, the operator $\mathscr{D}(z_t)$ is stable if the condition (9) holds. Note that the condition (9) is equivalent to $\rho(|A_1|) < 1$. Therefore, by Theorem 9.8.1 (pp. 292–293) of Hale and Lunel [9] with the stable operator $\mathscr{D}(z_t)$ and (18), we conclude that system (1) and (5) are both asymptotically stable.

Here, noting $\overline{\Gamma} \geqslant 0$ and from (17), we have

$$z^{\mathrm{T}}(t)\overline{\Gamma} z(t) < \dot{V}. \tag{19}$$

Integrating both sides of (19) from 0 to t_f leads to

$$\int_0^{t_f} z^{\mathrm{T}}(t)\overline{\Gamma} z(t)\,\mathrm{d}t < V(0) - V(t_f). \tag{20}$$

Since the closed-loop system (5) is asymptotically stable, $V(\infty) \to 0$ when $t_f \to \infty$. Hence we have

$$J = \int_0^{\infty} z^{\mathrm{T}}(t)\overline{\Gamma} z(t)\,\mathrm{d}t < V(0). \tag{21}$$

On the other hand, in the matrix $\bar{\Sigma}$, the matrices $P > 0$ and the controller parameters A_c, B_c and C_c, which included in the matrix $\overline{A}$ and $\overline{B}$, are unknown and occur in nonlinear fashion. Hence, the inequality $\overline{\Sigma} < 0$ cannot be considered as an linear matrix inequality problem, which can be solved by various convex optimization algorithms. In the following, we will use a method of changing variables such that the inequality can be solved as convex optimization algorithms [12].

First, partition the matrix P and its inverse as

$$P = \begin{bmatrix} S & N \\ N^{\mathrm{T}} & U_1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y & M \\ M^{\mathrm{T}} & U_2 \end{bmatrix}, \tag{22}$$

where $S, Y \in \mathbb{R}^{n\times n}$ are positive-definite matrices, and M and N are invertible matrices. Note that the identity $P^{-1}P = I$ gives that

$$MN^{\mathrm{T}} = I - YS. \tag{23}$$

Define

$$F_1 = \begin{bmatrix} Y & I \\ M^{\mathrm{T}} & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} I & S \\ 0 & N^{\mathrm{T}} \end{bmatrix}. \tag{24}$$

Then, it follows that

$$PF_1 = F_2, \quad F_1^{\mathrm{T}} P F_1 = F_1^{\mathrm{T}} F_2 = \begin{bmatrix} Y & I \\ I & S \end{bmatrix} > 0. \tag{25}$$

Now, postmultiplying and premultiplying the matrix inequality, $\overline{\Sigma} < 0$, by the matrix $\mathrm{diag}\{F_1^{\mathrm{T}}, F_1^{\mathrm{T}}, F_1^{\mathrm{T}}, F_1^{\mathrm{T}}\}$ and by its transpose, respectively, gives

$$\begin{bmatrix} \begin{pmatrix} F_2^{\mathrm{T}}\overline{A}F_1 + F_1^{\mathrm{T}}\overline{A}^{\mathrm{T}}F_2 \\ +hF_1^{\mathrm{T}}\overline{A}_1^{\mathrm{T}}W_1\overline{A}_1F_1 + F_1^{\mathrm{T}}\overline{A}_2^{\mathrm{T}}W_2\overline{A}_2F_1 \\ +F_1^{\mathrm{T}}\overline{B}^{\mathrm{T}}W_3\overline{B}F_1 + F_1^{\mathrm{T}}\overline{\Gamma}F_1 \end{pmatrix} & -F_1^{\mathrm{T}}\overline{A}^{\mathrm{T}}F_2 & hF_1^{\mathrm{T}}\overline{A}^{\mathrm{T}}F_2 & hF_1^{\mathrm{T}}F_2 \\ \star & -hF_1^{\mathrm{T}}W_1F_1 & 0 & -hF_1^{\mathrm{T}}F_2 \\ \star & \star & -F_1^{\mathrm{T}}W_3F_1 & h^2F_1^{\mathrm{T}}F_2 \\ \star & \star & \star & -F_1^{\mathrm{T}}W_2F_1 \end{bmatrix} < 0. \tag{26}$$

Here, define the matrices W_1, W_2 and W_3 as

$$W_1 = \mathrm{diag}\{\epsilon I, Q\}, \quad W_2 = \mathrm{diag}\{\eta I, Q\}, \quad W_3 = \mathrm{diag}\{\delta I, Q\},$$

where δ, η and ϵ are positive scalars, and Q is the positive-definite matrix to be chosen later.

By utilizing the relation (22)–(25), it can be easily obtained that the inequality (26) is equivalent to

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (1,7) & (1,8) \\ \star & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) & (2,7) & (2,8) \\ \star & \star & (3,3) & -h\epsilon Y & 0 & 0 & (3,7) & (3,8) \\ \star & \star & \star & -h\epsilon I & 0 & 0 & (4,7) & (4,8) \\ \star & \star & \star & \star & (5,5) & -\delta Y & (5,7) & (5,8) \\ \star & \star & \star & \star & \star & -\delta I & (6,7) & (6,8) \\ \star & \star & \star & \star & \star & \star & (7,7) & -\eta Y \\ \star & \star & \star & \star & \star & \star & \star & -\eta I \end{bmatrix} < 0, \tag{27}$$

where

$$\begin{aligned}
(1,1) &= AY + YA^{\mathrm{T}} + BC_cM^{\mathrm{T}} + MC_c^{\mathrm{T}}B^{\mathrm{T}} + \epsilon hYA_1^{\mathrm{T}}A_1Y + \delta MC_c^{\mathrm{T}}B_1^{\mathrm{T}}B_1C_cM^{\mathrm{T}} + \eta YA_2^{\mathrm{T}}A_2Y + Y\Gamma_1Y + MC_c^{\mathrm{T}}\Gamma_2C_cM^{\mathrm{T}},\\
(1,2) &= A + YA^{\mathrm{T}}S + MC_c^{\mathrm{T}}B^{\mathrm{T}}S + YC^{\mathrm{T}}B_c^{\mathrm{T}}N^{\mathrm{T}} + MA_c^{\mathrm{T}}N^{\mathrm{T}} + \epsilon hYA_1^{\mathrm{T}}A_1 + \eta YA_2^{\mathrm{T}}A_2 + Y\Gamma_1,\\
(1,3) &= -(YA^{\mathrm{T}} + MC_c^{\mathrm{T}}B^{\mathrm{T}}),\\
(1,4) &= -(YA^{\mathrm{T}}S + YC^{\mathrm{T}}B_c^{\mathrm{T}}N^{\mathrm{T}} + MC_c^{\mathrm{T}}B^{\mathrm{T}}S + MA_c^{\mathrm{T}}N^{\mathrm{T}}),\\
(1,5) &= -h\cdot(1,3), \quad (1,6) = -h\cdot(1,4),\\
(1,7) &= hY, \quad (1,8) = hI,\\
(2,2) &= SA + NB_cC + A^{\mathrm{T}}S + C^{\mathrm{T}}B_c^{\mathrm{T}}N^{\mathrm{T}} + \epsilon hA_1^{\mathrm{T}}A_1 + \eta A_2^{\mathrm{T}}A_2 + \Gamma_1,\\
(2,3) &= -A^{\mathrm{T}}, \quad (2,4) = -(A^{\mathrm{T}}S + C^{\mathrm{T}}B_c^{\mathrm{T}}N^{\mathrm{T}}),\\
(2,5) &= hA^{\mathrm{T}}, \quad (2,6) = h(A^{\mathrm{T}}S + C^{\mathrm{T}}B_c^{\mathrm{T}}N^{\mathrm{T}}),\\
(2,7) &= hI, \quad (2,8) = hS,\\
(3,3) &= -h(\epsilon YY + MQM^{\mathrm{T}}),\\
(3,7) &= -hY, \quad (3,8) = -hI,\\
(4,7) &= -hI, \quad (4,8) = -hS,\\
(5,5) &= -(\delta YY + MQM^{\mathrm{T}}),\\
(5,7) &= h^2Y, \quad (5,8) = h^2I,\\
(6,7) &= h^2I, \quad (6,8) = h^2S,\\
(7,7) &= -\eta YY - MQM^{\mathrm{T}}.
\end{aligned}$$

By defining a new set of variables as follows:

$$\begin{aligned}
X &= MQM^{\mathrm{T}},\\
\widehat{A} &= SAY + SB\widehat{C} + \widehat{B}CY + NA_cM^{\mathrm{T}},\\
\widehat{B} &= NB_c,\\
\widehat{C} &= C_cM^{\mathrm{T}},
\end{aligned} \tag{28}$$

the inequality (27) is simplified to the following inequality:

$$\Pi = \begin{bmatrix} \Pi_1 & \Omega_2 & \Omega_3 & -\widehat{A}^{\mathrm{T}} & -h\cdot\Omega_3 & h\widehat{A}^{\mathrm{T}} & hY & hI \\ \star & \Omega_4 & -A^{\mathrm{T}} & \Omega_5 & hA^{\mathrm{T}} & -h\cdot\Omega_5 & hI & hS \\ \star & \star & -hX-\epsilon hYY & -\epsilon hY & 0 & 0 & -hY & -hI \\ \star & \star & \star & -\epsilon hI & 0 & 0 & -hI & -hS \\ \star & \star & \star & \star & -X-\delta YY & -\delta Y & h^2Y & h^2I \\ \star & \star & \star & \star & \star & -\delta I & h^2I & h^2S \\ \star & \star & \star & \star & \star & \star & -X-\eta YY & -\eta Y \\ \star & \star & \star & \star & \star & \star & \star & -\eta I \end{bmatrix} < 0, \tag{29}$$

where $\Pi_1 = \Omega_1 + \epsilon hYA_1^{\mathrm{T}}A_1Y + \delta\widehat{C}^{\mathrm{T}}B_1^{\mathrm{T}}B_1\widehat{C} + \eta YA_2^{\mathrm{T}}A_2Y + Y\Gamma_1Y + \widehat{C}^{\mathrm{T}}\Gamma_2\widehat{C}$, and Ω_i (i = 1,2,...,5) are defined in (12).

Note that the inequality $\Pi < 0$ holds if

$$\Pi + \mathrm{diag}\{0, 0, \epsilon hYY, 0, \delta YY, 0, \eta YY, 0\} < 0. \tag{30}$$

By Fact 2 (Schur complement), the inequality (30) is equivalent to the inequality (10). By simple calculation, the upper bound of cost function J, $V(0)$ given in (21), is equivalent to (13). This completes our proof. □

Remark 6. The problem of Theorem 5 is to determine whether the problem is feasible or not. It is called the feasibility problem. The solutions of the problem can be found by solving generalized eigenvalue problem in S, Y, X, $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, δ, η, and ϵ, which is a quasiconvex optimization problem. Note that a locally optimal point of a quasiconvex optimization problem with strictly quasiconvex objective is globally optimal [10]. Various efficient convex optimization algorithms can be used to check whether the matrix inequalities (10) is feasible. In this paper, in order to solve the matrix inequality, we utilize Matlab's LMI Control Toolbox [13], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [10].

Theorem 5 presents a method of designing a state feedback guaranteed cost controller. The following theorem presents a method of selecting a controller minimizing the upper bound of the guaranteed cost (13).

Theorem 7. *Consider the system* (5) *with cost function* (3). *If the following optimization problem*

$$\min_{S>0,Y>0,X>0,\eta>0,\epsilon>0,\delta>0,\beta>0,\Psi_1>0,\Psi_2>0} \{\beta + \mathrm{Tr}(\Psi_1) + \mathrm{Tr}(\Psi_2)\}, \tag{31}$$

subject to

(i) *matrix inequality* (10)

(ii) $\begin{bmatrix} -\beta & \mathscr{X}^{\mathrm{T}}S \\ \star & -S \end{bmatrix} < 0,$

(iii) $\begin{bmatrix} -\Psi_1 & \epsilon\mathscr{R}_1^{\mathrm{T}}A_1^{\mathrm{T}} \\ \star & -\epsilon I \end{bmatrix} < 0,$

(iv) $\begin{bmatrix} -\Psi_2 & \eta\mathscr{R}_2^{\mathrm{T}}A_2^{\mathrm{T}} \\ \star & -\eta I \end{bmatrix} < 0,$

has a positive solution set ($S, Y, X, \eta, \delta, \epsilon, \beta, \Psi_1, \Psi_2$), *then the control law* (4) *is an optimal guaranteed cost control law which ensures the minimization of the guaranteed cost* (13) *for the neutral system* (5), *where*

$$\mathscr{X} = x(0) - A_1x(-h),$$

$$\mathscr{R}_1\mathscr{R}_1^{\mathrm{T}} = \int_{-h}^{0} x(s)x^{\mathrm{T}}(s)\,\mathrm{d}s,$$

$$\mathscr{R}_2\mathscr{R}_2^{\mathrm{T}} = (1/h)\int_{-h}^{0} (s+h)x(s)x^{\mathrm{T}}(s)\,\mathrm{d}s. \tag{32}$$

Proof. By Theorem 5, (i) in (31) is clear. Also, it follows from Fact 2 that (ii), (iii), and (iv) in (31) are equivalent to $(x(0) - A_1x(-h))^{\mathrm{T}}\ S(x(0) - A_1x(-h)) < \beta$, $\epsilon\mathscr{R}_1^{\mathrm{T}}A_1^{\mathrm{T}}A_1\mathscr{R}_1 < \Psi_1$, and $\mathscr{R}_2^{\mathrm{T}}A_2^{\mathrm{T}}A_2\mathscr{R}_2 < \Psi_2$, respectively. On the other hand,

$$\begin{aligned}\epsilon h\int_{-h}^{0}x^{\mathrm{T}}(s)A_1^{\mathrm{T}}A_1x(s)\,\mathrm{d}s &= \epsilon h\int_{-h}^{0}\mathrm{Tr}(x^{\mathrm{T}}(s)A_1^{\mathrm{T}}A_1x(s))\,\mathrm{d}s = \mathrm{Tr}\left(\epsilon A_1^{\mathrm{T}}A_1\int_{-h}^{0}x(s)x^{\mathrm{T}}(s)\,\mathrm{d}s\right) = \mathrm{Tr}(\epsilon A_1^{\mathrm{T}}A_1\mathscr{R}_1\mathscr{R}_1^{\mathrm{T}})\\ &= \mathrm{Tr}(\epsilon\mathscr{R}_1^{\mathrm{T}}A_1^{\mathrm{T}}A_1\mathscr{R}_1) < \mathrm{Tr}(\Psi_1),\end{aligned}$$

and

$$\begin{aligned}(\eta/h)\int_{-h}^{0}(s+h)x^{\mathrm{T}}(s)A_2^{\mathrm{T}}A_2x(s)\,\mathrm{d}s &= (\eta/h)\int_{-h}^{0}\mathrm{Tr}\big((s+h)x^{\mathrm{T}}(s)A_2^{\mathrm{T}}A_2x(s)\big)\,\mathrm{d}s\\ &= (\eta/h)\mathrm{Tr}\left(A_2^{\mathrm{T}}A_2\int_{-h}^{0}(s+h)x(s)x^{\mathrm{T}}(s)\,\mathrm{d}s\right) = \eta\cdot\mathrm{Tr}(A_2^{\mathrm{T}}A_2\mathscr{R}_2\mathscr{R}_2^{\mathrm{T}})\\ &= \eta\mathrm{Tr}(\mathscr{R}_2^{\mathrm{T}}A_2^{\mathrm{T}}A_2\mathscr{R}_2) < \mathrm{Tr}(\Psi_2).\end{aligned}$$

Hence, it follows from (13) that

$$J^* < \beta + \mathrm{Tr}(\Psi_1) + \mathrm{Tr}(\Psi_2).$$

Thus, the minimization of $\beta + \mathrm{Tr}(\Psi_1) + \mathrm{Tr}(\Psi_2)$ implies the minimization of the guaranteed cost for the system (5). In light of Remark 6, the convexity of this optimization problem ensures that a global optimum, when it exists, is reachable. □

Remark 8. Given any solution of the matrix inequalities (10) and (11) and (31) in Theorems 5 and 7, a corresponding controller of the form (4) will be constructed as follows:

- Using the solution X and selecting any positive-definite matrix Q, compute the invertible matrices M satisfying the relation $X = MQM^{\mathrm{T}}$.
- Using the matrix M, computer the invertible matrix N satisfying (23).
- Utilizing the matrices M and N obtained above, solve the system of equations (28) for B_c,C_c and A_c (in this order).

Let us consider the following numerical example to illustrate the design procedure.

Example 9. Consider the following neutral delay-differential system of the form:

$$\begin{cases}\dot{x}(t) = Ax(t) + A_1\dot{x}(t-h) + A_2\int_{t-h}^{t}x(s)\,\mathrm{d}s + B_0u(t) + B_1u(t-h),\\ y(t) = Cx(t),\end{cases} \tag{33}$$

where

$$A = \begin{bmatrix}2 & 0.1\\ 0.5 & -1\end{bmatrix},\quad A_1 = \begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix},\quad A_2 = \begin{bmatrix}0.1 & 0.2\\ 0.2 & 0.1\end{bmatrix},$$
$$B_0 = \begin{bmatrix}0\\ 1\end{bmatrix},\quad B_1 = \begin{bmatrix}2\\ 1\end{bmatrix},\quad C = [1\quad 0],\quad h = 1,$$

the initial condition of the system is as follows:

$$\begin{cases}x(t) = [-\mathrm{e}^{t}\ \ \mathrm{e}^{-t}]^{\mathrm{T}},\\ u(t) = 0,\end{cases}\text{for} -1 \leqslant t \leqslant 0.$$

Actually, when the control input is not forced to the system (33), i.e., $u(t) = 0$, the system is unstable since the states of the system go to infinity as $t \to \infty$.

To design dynamic output GCC for system (33), first, let us take a cost function (3) with the following weighting, $Q = I$ and $S = I$. Then, from the relations (32), we have

$$\mathscr{X} = \begin{bmatrix}-0.9264\\ 0.4563\end{bmatrix},\quad \mathscr{R}_1 = \begin{bmatrix}0.4761 & -0.4535\\ -0.4535 & 1.7288\end{bmatrix},\quad \mathscr{R}_2 = \begin{bmatrix}0.3881 & -0.3650\\ -0.3650 & 0.9819\end{bmatrix}. \tag{34}$$

Second, let us check the stability of the operator $\mathscr{D}(z_t)$ given in (9). Since $\rho(|A_1|) = 0.2 < 1$, the operator is stable.

Third, by applying Theorem 5 to system (33) and checking the feasibility of inequalities (10) and (11), we can find that the inequalities are feasible and obtain a possible solution set of the inequalities:

$$S = \begin{bmatrix} 74.9097 & 2.4692 \\ 2.4692 & 71.4895 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.2187 & 0.1958 \\ 0.1958 & 1.9191 \end{bmatrix},$$

$$X = \begin{bmatrix} 151.7547 & 0.5520 \\ 0.5520 & 151.3207 \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} -1.0019 & -0.0622 \\ -0.6440 & -1.5664 \end{bmatrix},$$

$$\widehat{B} = \begin{bmatrix} -221.3691 \\ -44.9023 \end{bmatrix}, \quad \widehat{C} = [-0.6689 \quad -0.6447],$$

$$\epsilon = 183.3262, \quad \delta = 207.2713, \quad \eta = 184.0131.$$

Finally, let us choose $Q = 10^5 \times I$. Then in light of Remark 8, two invertible matrices M, N are

$$M = \begin{bmatrix} 0.0390 & 0.0001 \\ 0.0001 & 0.0389 \end{bmatrix}, \quad N = 10^3 \times \begin{bmatrix} -0.4064 & -0.4982 \\ -0.3669 & -3.5129 \end{bmatrix},$$

and the corresponding positive-definite matrix P is

$$P = 10^5 \times \begin{bmatrix} 0.0007 & 0.0000 & -0.0041 & -0.0050 \\ 0.0000 & 0.0007 & -0.0037 & -0.0351 \\ -0.0041 & -0.0037 & 0.0409 & 0.2014 \\ -0.0050 & -0.0351 & 0.2014 & 1.7577 \end{bmatrix}.$$

Then, a possible stabilizing dynamic output feedback controller for the system (33) is as follows:

$$A_c = \begin{bmatrix} -7.2962 & -5.1396 \\ -0.0620 & -1.1547 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.6067 \\ -0.0506 \end{bmatrix}, \quad C_c = [-17.1403 \quad -16.5429],$$

and using (34) and solution variables S, ϵ, η, its corresponding upper bound (13) of cost function is

$$J \leqslant 109.0345.$$

By the way, by applying the optimization problem given in Theorem 7 to the system (33), we can obtain the minimal upper bound of the cost function as

$$J^* = 5.492.$$

3. Conclusions

In this article, we have proposed a design method of dynamic output GCC for a class of neutral delay-differential systems with delay in control input. The proposed method guarantees asymptotic stability of the closed-loop system and minimization of upper bound of given cost function. Also, the variables of the controller can be found by solving matrix inequalities. A numerical example is shown to illustrate the design procedure.

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