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Improved delay-dependent stability criterion for neural networks with time-varying delays

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ABSTRACT

In this Letter, the problem of stability analysis for neural networks with time-varying delays is considered. By constructing a new Lyapunov functional, a new delay-dependent stability criterion for the network is established in terms of LMIs (linear matrix inequalities) which can be easily solved by various convex optimization algorithms. Two numerical examples are included to show the effectiveness of proposed criterion.

1. Introduction

Since neural networks have been successfully applied to various fields such as pattern recognition, associative memories, signal processing, fixed-point computations, and so on, the stability analysis of neural networks has been extensively investigated [1–17]. On the other hand, time delays are frequently encountered in neural networks due to the finite switching speed of amplifiers and the inherent communication of neurons [3–5]. It is well known the existence of time delay may cause divergence, oscillation, and even instability. Therefore, considerable efforts have been devoted to stability analysis of neural networks with time delays since the applications of neural networks with time delays heavily depend on the dynamic behavior of the equilibrium point [6–17].

In deriving the delay-dependent stability criterion of dynamic system with time delays, the main concern is to enlarge the feasibility region of stability criteria or to get the maximum allowable bound of time delays for guaranteeing the stability. In general, the delay-dependent criterion is less conservative than delay-independent one when the value of the size of delays are small. Therefore, how to choose Lyapunov–Krasovskii functionals and derive the stability condition by calculating the upper bounds of time derivative of Lyapunov–Krasovskii functionals play key roles to increase the maximum allowable bound of time delays. In this regard, Hua et al. [15] proposed a new Lyapunov–Krasovskii functional and employed free-weighting matrices to enlarge the delay bounds of neural networks with time-varying delays. Recently, Kwon et al. [16] investigated robust stability criterion for neural networks with interval time-varying delays. Very recently, Chen and Wu [17] presented an improved stability criterion by constructing the augmented Lyapunov functionals and resorting to the new technique for estimating the upper bound of the derivative of Lyapunov functionals.

In this Letter, we revisit the problem of delay-dependent stability analysis of neural networks with time-varying delays. By constructing a new Lyapunov–Krasovskii functional which fractions delay interval and employing different free-weighting matrices in the upper bounds of integral terms, a novel delay-dependent stability criterion is derived in terms of LMIs which can be solved efficiently by using well-

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known interior-point algorithms [18]. Two numerical examples are shown to support that our results are less conservative than those of the existing literature.

Notation

$\mathcal{R}^n$ is the n-dimensional Euclidean space, $\mathcal{R}^{m\times n}$ denotes the set of $m \times n$ real matrix. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices X and Y, the notation $X > Y$ (respectively, $X \geqslant Y$) means that the matrix $X - Y$ is positive definite, (respectively, nonnegative). diag$\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix.

2. Problem statements

Consider the following uncertain neural networks with discrete time-varying delays:

$$\dot{y}(t) = -Ay(t) + W_0 g\big(y(t)\big) + W_1 g\big(y\big(t-h(t)\big)\big) + b, \tag{1}$$

where $y(t) = [y_1(t), \ldots, y_n(t)]^T \in \mathcal{R}^n$ is the neuron state vector, n denotes the number of neurons in a neural network, $g(y(t)) = [g_1(y_1(t)), \ldots, g_n(y_n(t))]^T \in \mathcal{R}^n$ denotes the neuron activation function, $g(y(t-h(t))) = [g_1(y_1(t-h(t))), \ldots, g_n(y_n(t-h(t)))]^T \in \mathcal{R}^n$, $A = \mathrm{diag}\{a_i\} \in \mathcal{R}^{n\times n}$ is a positive diagonal matrix, $W_0 = (w_{ij}^0)_{n\times n} \in R^{n\times n}$, and $W_1 = (w_{ij}^1)_{n\times n} \in \mathcal{R}^{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons, $b = [b_1, b_2, \ldots, b_n]^T \in \mathcal{R}^n$ means a constant input vector.

The delays, $h(t)$, are time-varying continuous functions that satisfies

$$0 \leqslant h(t) \leqslant h_U, \quad \dot{h}(t) \leqslant h_D, \tag{2}$$

where h_U is a positive scalar and h_D is any constant one.

The activation functions, $g_i(y_i(t))$, $i = 1, \ldots, n$, are assumed to be nondecreasing, bounded and globally Lipschiz; that is,

$$0 \leqslant \frac{g_i(\xi_i) - g_j(\xi_j)}{\xi_i - \xi_j} \leqslant l_i, \quad \xi_i, \xi_j \in \mathcal{R}, \quad \xi_i \neq \xi_j, \quad i = 1, \ldots, n, \tag{3}$$

where l_i, $i = 1, \ldots, n$, are positive constants.

Note that by using the Brouwer's fixed-point theorem [6], it can be easily proven that there exists at least one equilibrium point for Eq. (1).

For simplicity, in stability analysis of the system (1), the equilibrium point $y^* = [y_1^*, \ldots, y_n^*]^T$ is shifted to the origin by utilizing the transformation $x(\cdot) = y(\cdot) - y^*$, which leads the system (1) to the following form:

$$\dot{x}(t) = -Ax(t) + W_0 f\big(x(t)\big) + W_1 f\big(x\big(t-h(t)\big)\big), \tag{4}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathcal{R}^n$ is the state vector of the transformed system,

Note that $f(x(t)) = [f_1(x(t)), \ldots, f_n(x(t))]^T$ and $f_j(x_j(t)) = g_j(x_j(t) + y_j^*) - g_j(y_j^*)$ with $f_j(0) = 0$ $(j = 1, \ldots, n)$.

From Eq. (3), $f_j(\cdot)$ satisfies the following condition:

$$0 \leqslant \frac{f_j(\xi_j)}{\xi_j} \leqslant l_j, \quad \forall \xi_j \neq 0, \quad j = 1, \ldots, n, \tag{5}$$

which is equivalent to

$$f_j(\xi_j)\big[f_j(\xi_j) - l_j\xi_j\big] \leqslant 0, \quad f_j(0) = 0, \quad j = 1, \ldots, n. \tag{6}$$

The objective of this Letter is to investigate the delay dependent stability analysis of system (4) which will be conducted in Section 3.

Before deriving our main results, we state the following facts, and lemma.

Fact 1 *(Schur complement). Given constant symmetric matrices Σ_1, Σ_2, Σ_3, where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if*

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

Fact 2. *For any real vectors a, b and any matrix $Q > 0$ with appropriate dimensions, it follows that:*

$$\pm 2a^T b \leqslant a^T Q a + b^T Q^{-1} b.$$

To derive a less conservative stability criterion, we will use the following lemma which will be used in deriving upper bounds of integral terms.

Lemma 1. *For any scalar $h(t) \geqslant 0$, and any constant matrix $Q \in \mathcal{R}^{n\times n}$, $Q = Q^T > 0$, the following inequality holds:*

$$-\int_{t-h(t)}^{t} \dot{x}^T(s) Q \dot{x}(s)\, ds \leqslant h(t)\zeta^T(t)\mathcal{X} Q^{-1} \mathcal{X}^T \zeta(t) + 2\zeta^T(t)\big[x(t) - x\big(t-h(t)\big)\big], \tag{7}$$

where

$$\zeta^T(t) = \left[x^T(t) \quad x^T\big(t-h(t)\big) \quad x^T\left(t - \frac{h_U}{2}\right) \quad x^T(t-h_U) \quad \dot{x}^T(t) \quad f\big(x(t)\big) \quad f\big(x\big(t-h(t)\big)\big)\right], \tag{8}$$

and $\mathcal{X}$ is free-weighting matrix with appropriate dimensions.

Proof. By using Fact 2, the following inequality holds:

$$-2\int_{t-h(t)}^{t}\left(\mathcal{X}^T\zeta(t)\right)^T\dot{x}(s)\,ds \leqslant \int_{t-h(t)}^{t}\left[\zeta^T(t)\mathcal{X}Q^{-1}\mathcal{X}^T\zeta(t)+\dot{x}^T(s)Q\dot{x}(s)\right]ds. \tag{9}$$

From Eq. (9), we obtain

$$\begin{aligned}
-\int_{t-h(t)}^{t}\dot{x}^T(s)Q\dot{x}(s)\,ds &\leqslant \int_{t-h(t)}^{t}\zeta^T(t)\mathcal{X}Q^{-1}\mathcal{X}^T\zeta(t)\,ds+2\int_{t-h(t)}^{t}\left(\mathcal{X}^T\zeta(t)\right)^T\dot{x}(s)\,ds\\
&= h(t)\zeta^T(t)\mathcal{X}Q^{-1}\mathcal{X}^T\zeta(t)\,ds+2\zeta^T(t)\mathcal{X}\left[x(t)-x\big(t-h(t)\big)\right].
\end{aligned} \tag{10}$$

This completes the proof. □

3. Main results

In this section, a new delay-dependent stability criterion for neural networks with time-varying delays (4) will be derived. Before introducing main results, the notations of several matrices are defined for simplicity:

$$\begin{aligned}
&\Sigma=[\Sigma_{(i,j)}]\quad (i,j=1,\dots,7),\\
&\Sigma_{(1,1)}=G_{11}+N_{11}-P_1A-A^TP_1^T,\qquad \Sigma_{(1,2)}=-(1-h_D)N_{11},\qquad \Sigma_{(1,3)}=G_{12},\\
&\Sigma_{(1,4)}=0,\qquad \Sigma_{(1,5)}=R_1-P_1-A^TP_2^T,\qquad \Sigma_{(1,6)}=N_{12}+LH_1+P_1W_0,\\
&\Sigma_{(1,7)}=P_1W_1,\qquad \Sigma_{(2,2)}=0,\qquad \Sigma_{(2,3)}=0,\qquad \Sigma_{(2,4)}=0,\qquad \Sigma_{(2,5)}=0,\qquad \Sigma_{(2,6)}=0,\\
&\Sigma_{(2,7)}=-(1-h_D)N_{12}+LH_2,\qquad \Sigma_{(3,3)}=G_{22}-G_{11},\qquad \Sigma_{(3,4)}=-G_{12},\\
&\Sigma_{(3,5)}=0,\qquad \Sigma_{(3,6)}=0,\qquad \Sigma_{(3,7)}=0,\qquad \Sigma_{(4,4)}=-G_{22},\qquad \Sigma_{(4,5)}=0,\qquad \Sigma_{(4,6)}=0,\\
&\Sigma_{(4,7)}=0,\qquad \Sigma_{(5,5)}=\frac{h_U}{2}(Q_1+Q_2)-P_2-P_2^T,\qquad \Sigma_{(5,6)}=K+P_2W_0,\\
&\Sigma_{(5,7)}=P_2W_1,\qquad \Sigma_{(6,6)}=N_{22}-2H_1,\qquad \Sigma_{(6,7)}=0,\qquad \Sigma_{(7,7)}=-(1-h_D)N_{22}-2H_2,\\
&\mathcal{X}=\begin{bmatrix}X_1^T & X_2^T & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T,\qquad \mathcal{Y}=\begin{bmatrix}0 & Y_1^T & Y_2^T & 0 & 0 & 0 & 0\end{bmatrix}^T,\qquad \mathcal{Z}=\begin{bmatrix}0 & 0 & Z_1^T & Z_2^T & 0 & 0 & 0\end{bmatrix}^T,\\
&\Upsilon=\begin{bmatrix}\mathcal{X} & -\mathcal{X}+\mathcal{Y} & -\mathcal{Y}+\mathcal{Z} & -\mathcal{Z} & 0 & 0 & 0\end{bmatrix},\\
&\bar{\mathcal{X}}=\begin{bmatrix}\bar{X}_1^T & 0 & \bar{X}_2^T & 0 & 0 & 0 & 0\end{bmatrix}^T,\qquad \bar{\mathcal{Y}}=\begin{bmatrix}0 & \bar{Y}_1^T & \bar{Y}_2^T & 0 & 0 & 0 & 0\end{bmatrix}^T,\qquad \bar{\mathcal{Z}}=\begin{bmatrix}0 & \bar{Z}_1^T & 0 & \bar{Z}_2^T & 0 & 0 & 0\end{bmatrix}^T,\\
&\bar{\Upsilon}=\begin{bmatrix}\bar{\mathcal{X}} & -\bar{\mathcal{Y}}+\bar{\mathcal{Z}} & -\bar{\mathcal{X}}+\bar{\mathcal{Y}} & -\bar{\mathcal{Z}} & 0 & 0 & 0\end{bmatrix}.
\end{aligned} \tag{11}$$

Now, we have the following theorem.

Theorem 1. *For given scalars $h_U>0$, h_D, and $L=\mathrm{diag}\{l_1,\dots,l_n\}$, the equilibrium point of system (1) is globally asymptotically stable for $0\leqslant h(t)\leqslant h_U$ and $\dot{h}(t)\leqslant h_D$ if there exist positive diagonal matrices $K=\mathrm{diag}\{k_1,\dots,k_n\}$, $H_1=\mathrm{diag}\{h_{1i},\dots,h_{1n}\}$, $H_2=\mathrm{diag}\{h_{2i},\dots,h_{2n}\}$, positive matrices $R_1>0$, $\begin{bmatrix}N_{11} & N_{12}\\ \star & N_{22}\end{bmatrix}>0$, $\begin{bmatrix}G_{11} & G_{12}\\ \star & G_{22}\end{bmatrix}>0$, Q_i $(i=1,2)>0$, and any matrices P_1, P_2, X_i, Y_i, Z_i, $\bar{X}_i$, $\bar{Y}_i$, $\bar{Z}_i$ $(i=1,2)$ satisfying the following LMIs:*

$$\begin{bmatrix}\Sigma+\Upsilon+\Upsilon^T & \frac{h_U}{2}\mathcal{Y} & \frac{h_U}{2}\mathcal{Z}\\ \star & -\frac{h_U}{2}Q_1 & 0\\ \star & \star & -\frac{h_U}{2}Q_2\end{bmatrix}<0, \tag{12}$$

$$\begin{bmatrix}\Sigma+\Upsilon+\Upsilon^T & \frac{h_U}{2}\mathcal{X} & \frac{h_U}{2}\mathcal{Z}\\ \star & -\frac{h_U}{2}Q_1 & 0\\ \star & \star & -\frac{h_U}{2}Q_2\end{bmatrix}<0, \tag{13}$$

$$\begin{bmatrix}\Sigma+\bar{\Upsilon}+\bar{\Upsilon}^T & \frac{h_U}{2}\bar{\mathcal{X}} & \frac{h_U}{2}\bar{\mathcal{Z}}\\ \star & -\frac{h_U}{2}Q_1 & 0\\ \star & \star & -\frac{h_U}{2}Q_2\end{bmatrix}<0, \tag{14}$$

$$\begin{bmatrix}\Sigma+\bar{\Upsilon}+\bar{\Upsilon}^T & \frac{h_U}{2}\bar{\mathcal{X}} & \frac{h_U}{2}\bar{\mathcal{Y}}\\ \star & -\frac{h_U}{2}Q_1 & 0\\ \star & \star & -\frac{h_U}{2}Q_2\end{bmatrix}<0. \tag{15}$$

Proof. For positive diagonal matrix $K=\mathrm{diag}\{k_1,\dots,k_n\}$, positive definite matrices R_1, $\begin{bmatrix}N_{11} & N_{12}\\ \star & N_{22}\end{bmatrix}$, $\begin{bmatrix}G_{11} & G_{12}\\ \star & G_{22}\end{bmatrix}$, Q_i $(i=1,2)$, let us consider the Lyapunov–Krasovskii functional candidate:

$$V=\sum_{i=1}^{4}V_i, \tag{16}$$

where

$$V_1 = x^T(t)R_1x(t) + 2\sum_{i=1}^{n} k_i \int_0^{x_i(t)} f_i(s)\,ds,$$

$$V_2 = \int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds,$$

$$V_3 = \int_{t-\frac{h_U}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{h_U}{2}) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-\frac{h_U}{2}) \end{bmatrix} ds,$$

$$V_4 = \int_{t-\frac{h_U}{2}}^{t} \int_s^t \dot{x}^T(u)Q_1\dot{x}(u)\,du\,ds + \int_{t-h_U}^{t-\frac{h_U}{2}} \int_s^t \dot{x}^T(u)Q_2\dot{x}(u)\,du\,ds. \tag{17}$$

The time-derivative of V_1 can be calculated as

$$\dot{V}_1 = 2x^T(t)R_1\dot{x}(t) + 2f^T\big(x(t)\big)K\dot{x}(t). \tag{18}$$

An upper bound of time-derivative of V_2 can be obtained as

$$\dot{V}_2 \leqslant \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} - (1-h_D) \begin{bmatrix} x(t-h(t)) \\ f(x(t-h(t))) \end{bmatrix}^T \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ f(x(t-h(t))) \end{bmatrix}. \tag{19}$$

Calculating $\dot{V}_3$ leads to

$$\dot{V}_3 = \begin{bmatrix} x(t) \\ x(t-\frac{h_U}{2}) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\frac{h_U}{2}) \end{bmatrix} - \begin{bmatrix} x(t-\frac{h_U}{2}) \\ x(t-h_U) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \begin{bmatrix} x(t-\frac{h_U}{2}) \\ x(t-h_U) \end{bmatrix}. \tag{20}$$

And $\dot{V}_4$ can be obtained as

$$\dot{V}_4 = \frac{h_U}{2}\dot{x}^T(t)Q_1\dot{x}(t) - \int_{t-\frac{h_U}{2}}^{t} \dot{x}^T(s)Q_1\dot{x}(s)\,ds + \frac{h_U}{2}\dot{x}^T(t)Q_2\dot{x}(t) - \int_{t-h_U}^{t-\frac{h_U}{2}} \dot{x}^T(s)Q_2\dot{x}(s)\,ds. \tag{21}$$

Using Lemma 1, two integral terms in $\dot{V}_4$ can be estimated as follows:

(i) When $0 \leqslant h(t) \leqslant \frac{h_U}{2}$, we have

$$\begin{aligned} -\int_{t-\frac{h_U}{2}}^{t} \dot{x}^T(s)Q_1\dot{x}(s)\,ds &= -\int_{t-h(t)}^{t} \dot{x}^T(s)Q_1\dot{x}(s)\,ds - \int_{t-\frac{h_U}{2}}^{t-h(t)} \dot{x}^T(s)Q_1\dot{x}(s)\,ds \\ &\leqslant h(t)\zeta^T(t)\mathcal{X}Q_1^{-1}\mathcal{X}^T\zeta(t) + 2\zeta^T(t)\mathcal{X}\big[x(t) - x\big(t-h(t)\big)\big] \\ &\quad + \left(\frac{h_U}{2} - h(t)\right)\zeta^T(t)\mathcal{Y}Q_1^{-1}\mathcal{Y}^T\zeta(t) + 2\zeta^T(t)\mathcal{Y}\left[x\big(t-h(t)\big) - x\left(t-\frac{h_U}{2}\right)\right], \end{aligned} \tag{22}$$

$$-\int_{t-h_U}^{t-\frac{h_U}{2}} \dot{x}^T(s)Q_2\dot{x}(s)\,ds \leqslant \frac{h_U}{2}\zeta^T(t)\mathcal{Z}Q_2^{-1}\mathcal{Z}^T\zeta(t) + 2\zeta^T(t)\mathcal{Z}\left[x\left(t-\frac{h_U}{2}\right) - x(t-h_U)\right]. \tag{23}$$

Remember that $\zeta(t)$ is defined in (8), and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are in Eq. (11).

As a tool of deriving less conservative results, we add the following zero equation with free variables P_1 and P_2 to be chosen as

$$0 = 2\big[x^T(t)P_1 + \dot{x}^T(t)P_2\big]\big[-\dot{x}(t) - Ax(t) + W_0 f\big(x(t)\big) + W_1 f\big(x\big(t-h(t)\big)\big)\big]. \tag{24}$$

Note that Eq. (5) means

$$f_j\big(x_j(t)\big)\big[f_j\big(x_j(t)\big) - l_j x_j(t)\big] \leqslant 0 \quad (j = 1,\ldots,n), \tag{25}$$

$$f_j\big(x_j\big(t-h(t)\big)\big)\big[f_j\big(x_j\big(t-h(t)\big)\big) - l_j x_j\big(t-h(t)\big)\big] \leqslant 0 \quad (j = 1,\ldots,n). \tag{26}$$

From the above two inequalities (25), (26), for any positive diagonal matrices $H_1 = \mathrm{diag}\{h_{1i},\ldots,h_{1n}\}$, and $H_2 = \mathrm{diag}\{h_{2i},\ldots,h_{2n}\}$, the following inequalities hold

$$\begin{aligned} 0 &\leqslant -2\sum_{j=1}^{n} h_{1j} f_j\big(x_j(t)\big)\big[f_j\big(x_j(t)\big) - l_j x_j(t)\big] - 2\sum_{j=1}^{n} h_{2j} f_j\big(x_j\big(t-h(t)\big)\big)\big[f_j\big(x_j\big(t-h(t)\big)\big) - l_j x_j\big(t-h(t)\big)\big] \\ &= 2x^T(t)LH_1 f\big(x(t)\big) - 2f^T\big(x(t)\big)H_1 f\big(x(t)\big) + 2x^T\big(t-h(t)\big)LH_2 f\big(x\big(t-h(t)\big)\big) - 2f^T\big(x\big(t-h(t)\big)\big)H_2 f\big(x\big(t-h(t)\big)\big). \end{aligned} \tag{27}$$

From (17)–(27) and by applying S-procedure [18], the $\dot{V} = \sum_{i=1}^{4} \dot{V}_i$ has a new upper bound as

$$\dot{V} \leqslant \zeta^T(t)\Omega_1\zeta(t), \tag{28}$$

where

$$\Omega_1 = \Sigma + \Upsilon + \Upsilon^T + h(t)\mathcal{X}Q_1^{-1}\mathcal{X}^T + \left(\frac{h_U}{2} - h(t)\right)\mathcal{Y}Q_1^{-1}\mathcal{Y}^T + \frac{h_U}{2}\mathcal{Z}Q_2^{-1}\mathcal{Z}^T, \tag{29}$$

and Σ, Υ are defined in (11).

Since

$$h(t)\mathcal{X}Q_1^{-1}\mathcal{X}^T + \left(\frac{h_U}{2} - h(t)\right)\mathcal{Y}Q_1^{-1}\mathcal{Y}^T + \frac{h_U}{2}\mathcal{Z}Q_2^{-1}\mathcal{Z}^T \tag{30}$$

is a convex combination of the matrices $\mathcal{X}Q_1^{-1}\mathcal{X}^T$, $\mathcal{Y}Q_1^{-1}\mathcal{Y}^T$, and $\mathcal{Z}Q_2^{-1}\mathcal{Z}^T$ on $h(t)$, $\Omega_1 < 0$ for $0 \leqslant h(t) \leqslant \frac{h_U}{2}$ can be handled by two corresponding boundary LMIs:

$$\Sigma + \Upsilon + \Upsilon^T + \frac{h_U}{2}\mathcal{Y}Q_1^{-1}\mathcal{Y}^T + \frac{h_U}{2}\mathcal{Z}Q_2^{-1}\mathcal{Z} < 0, \tag{31}$$

$$\Sigma + \Upsilon + \Upsilon^T + \frac{h_U}{2}\mathcal{X}Q_1^{-1}\mathcal{X}^T + \frac{h_U}{2}\mathcal{Z}Q_2^{-1}\mathcal{Z} < 0. \tag{32}$$

Using Fact 1, the inequalities (31) and (32) are equivalent to the LMIs (12) and (13), respectively.

(ii) When $\frac{h_U}{2} \leqslant h(t) \leqslant h_U$, we have upper bounds of integral terms in $\dot{V}_4(t)$ as

$$-\int_{t-\frac{h_U}{2}}^{t} \dot{x}^T(s)Q_1\dot{x}(s)\,ds \leqslant \frac{h_U}{2}\zeta^T(t)\bar{\mathcal{X}}Q_1^{-1}\bar{\mathcal{X}}^T\zeta(t) + 2\zeta^T(t)\bar{\mathcal{X}}\left[x(t) - x\left(t - \frac{h_U}{2}\right)\right], \tag{33}$$

$$\begin{aligned}
-\int_{t-h_U}^{t-\frac{h_U}{2}} \dot{x}^T(s)Q_2\dot{x}(s)\,ds &= -\int_{t-h(t)}^{t-\frac{h_U}{2}} \dot{x}^T(s)Q_2\dot{x}(s)\,ds - \int_{t-h_U}^{t-h(t)} \dot{x}^T(s)Q_2\dot{x}(s)\,ds \\
&\leqslant \left(h(t) - \frac{h_U}{2}\right)\zeta^T(t)\bar{\mathcal{Y}}Q_2^{-1}\bar{\mathcal{Y}}^T\zeta(t) + 2\zeta^T(t)\bar{\mathcal{Y}}\left[x\left(t - \frac{h_U}{2}\right) - x\big(t - h(t)\big)\right] \\
&\quad + \big(h_U - h(t)\big)\zeta^T(t)\bar{\mathcal{Z}}Q_2^{-1}\bar{\mathcal{Z}}^T\zeta(t) + 2\zeta^T(t)\bar{\mathcal{Z}}\big[x\big(t - h(t)\big) - x(t - h_U)\big],
\end{aligned} \tag{34}$$

where $\bar{\mathcal{X}}, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}$ are defined in Eq. (11).

From (17)–(21), (24)–(27), (33)–(34) and by applying S-procedure [18], $\dot{V} = \sum_{i=1}^{4} \dot{V}_i$ has a new upper bound as

$$\dot{V} \leqslant \zeta^T(t)\Omega_2\zeta(t), \tag{35}$$

where

$$\Omega_2 = \Sigma + \bar{\Upsilon} + \bar{\Upsilon}^T + \frac{h_U}{2}\bar{\mathcal{X}}Q_1^{-1}\bar{\mathcal{X}}^T + \left(h(t) - \frac{h_U}{2}\right)\bar{\mathcal{Y}}Q_2^{-1}\bar{\mathcal{Y}}^T + \big(h_U - h(t)\big)\bar{\mathcal{Z}}Q_2^{-1}\bar{\mathcal{Z}}^T, \tag{36}$$

and $\bar{\Upsilon}$ is defined in (11).

Using similar method in Eq. (30)–(32), $\Omega_2 < 0$ for $\frac{h_U}{2} \leqslant h(t) \leqslant h_U$ can be handled by two corresponding boundary LMIs:

$$\Sigma + \bar{\Upsilon} + \bar{\Upsilon}^T + \frac{h_U}{2}\bar{\mathcal{X}}Q_1^{-1}\bar{\mathcal{X}}^T + \frac{h_U}{2}\bar{\mathcal{Z}}Q_2^{-1}\bar{\mathcal{Z}}^T < 0, \tag{37}$$

$$\Sigma + \bar{\Upsilon} + \bar{\Upsilon}^T + \frac{h_U}{2}\bar{\mathcal{X}}Q_1^{-1}\bar{\mathcal{X}}^T + \frac{h_U}{2}\bar{\mathcal{Y}}Q_2^{-1}\bar{\mathcal{Y}}^T < 0. \tag{38}$$

By using Fact 1, the inequalities (37) and (38) are equivalent to LMIs (14) and (15), respectively. Therefore, if the LMIs (12)–(15) are satisfied, then the system (1) is guaranteed to be asymptotically stable. This completes our proof. □

Remark 1. In Eq. (17), the new Lyapunov functional which divides the delay interval $[0, h_U]$ into two ones $[0, \frac{h_U}{2}]$ and $[\frac{h_U}{2}, h_U]$ are proposed. Therefore, by taking the states $x(t - \frac{h_U}{2})$ and $x(t - h_U)$ as augmented variables simultaneously, the stability in Theorem 1 utilizes more information on state variables. And in deriving upper bounds of integral terms in $\dot{V}_3$, different free-weighting matrices are introduced in two different intervals $0 \leqslant h(t) \leqslant \frac{h_U}{2}$ and $\frac{h_U}{2} \leqslant h(t) \leqslant h_U$. These methods mentioned above are not considered in other literature and may lead to obtain an improved feasible region for delay-dependent stability criteria.

Remark 2. If we do not consider V_2 in Eq. (17), the stability criterion do not needs the information of time derivative of $h(t)$.

Table 1
Delay bounds h_U with different h_D (Example 1).

h_D	0.8	0.9	Unknown
Hua et al. [15]	1.2281	0.8636	0.8298
Kwon et al. [16]	1.6831	1.14934	1.0880
Chen et al. [17]	2.3534	1.6050	1.5103
Theorem 1	2.8854	1.9631	1.7810

Table 2
Delay bounds h_U with different h_D (Example 2).

h_D	0.1	0.5	0.9	Unknown
Hua et al. [15]	3.2775	2.1502	1.3164	1.2598
Kwon et al. [16]	3.2509	2.2098	1.5771	1.5369
Chen et al. [17]	3.3428	2.5421	2.0867	2.0389
Theorem 1	3.7525	2.7353	2.2760	2.1326

4. Numerical examples

Example 1. Consider the neural networks (4) with time-varying delays with the parameters

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \qquad L = \mathrm{diag}\{0.4, 0.8\}.$$

To the best of authors' knowledge, the best results [17] of delay bounds for guaranteeing stability of this system when h_D is 0.8, 0.9, and unknown are 2.3534, 1.6050, and 1.5103, respectively. However, by applying Theorem 1, the delay bounds when h_D is 0.8, 0.9, and unknown are 2.8854, 1.9631, and 1.7810, respectively. The improvement over the existing best result is over 22.6% for $h_D = 0.8$, over 22.3% for $h_D = 0.9$, and over 17.9% for unknown h_D. The detail comparison of our results with existing ones [15–17] is shown in Table 1. For Table 1, one can see that the result proposed in this Letter improves the existing results [15–17].

Example 2. Consider the following nominal delayed neural networks

$$\dot{x}(t) = -Ax(t) + W_0 g\big(x(t)\big) + W_1 g\big(x\big(t - h(t)\big)\big)$$

with

$$A = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}, \qquad W_0 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}, \qquad L = \mathrm{diag}\{0.1137, 0.1279, 0.7994, 0.2368\}.$$

Table 2 gives the comparison results on the maximum delay bound allowed via the methods in recent works and our new study. From Table 2, it can be seen that Theorem 1 gives larger delay bounds than very recent results in [15–17].

Example 3. Consider the following nominal delayed neural networks

$$\dot{x}(t) = -Ax(t) + W_0 g\big(x(t)\big) + W_1 g\big(x\big(t - h(t)\big)\big),$$

where

$$A = I, \qquad W_0 = \begin{bmatrix} 0 & 0.9 \\ -1 & -1 \end{bmatrix}, \qquad W_1 = 1.1 \times \begin{bmatrix} 0.6 & -1 \\ 1 & 0 \end{bmatrix},$$

and the activation function $g_i(x) = 0.5(|x+1| - |x-1|)$.

When $h(t) = h$ is constant, it is easy to verify that Theorem 1 in [9] is infeasible and Theorem 1 in [10] is feasible. And Theorem 1 in [13] is feasible for all $h > 0$. If we apply Theorem 1 to the above system with the same condition, it can be obtained that Theorem 1 in this Letter is also feasible for all $h > 0$. Note that the stability criteria in [9] and [10] are delay-independent and Theorem 1 in [13] and our stability criterion are delay-dependent. When h_D is 0.5, which means the considered delays are time-varying, all of the stability criteria [9] and [10] are infeasible and Theorem 1 in [13] is not applicable. However, Theorem 1 in this Letter provides delay bounds as 0.8977.

5. Conclusion

In this Letter, a new delay-dependent stability criterion for neural networks with time-varying delays is proposed. To obtain a less conservative stability criterion for neural network (1), a new Lyapunov–Krasovskii functional which fractions delay intervals has been proposed. A bounding technique of integral terms with free-weighting matrices in different delay intervals is utilized to reduce the conservatism of stability criterion. Numerical examples have been given to demonstrate the effectiveness of the presented criterion and its improvement over existing results.

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